# Integration by Parts

Derived from the “Product Rule”

<table>
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<th>Product Rule</th>
<th>Integrate both sides</th>
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<tr>
<td>( \frac{d}{dx}(f(x) \cdot g(x)) = g(x) \frac{d}{dx}f(x) + f(x) \frac{d}{dx}g(x) )</td>
<td>( \int \frac{d}{dx}(f(x) \cdot g(x)) , dx = f(x) \cdot g(x) )</td>
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<thead>
<tr>
<th>Obtain Standard Form</th>
<th>Recognize Standard form</th>
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<tr>
<td>( f(x) \frac{d}{dx}g(x) , dx = f(x) \cdot g(x) - \int g(x) \frac{d}{dx}f(x) , dx )</td>
<td>( f(x) \cdot g(x) = \int g(x) \frac{d}{dx}f(x) , dx + \int f(x) \frac{d}{dx}g(x) , dx )</td>
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</table>

\[ u \, dv = uv - \int v \, du \]
**Integration by Parts**

**Identifying the “Parts”**

\[
\int f(x) \, dg(x) = f(x) \cdot g(x) - \int g(x) \, df(x)
\]

\[
\int u \, dv = uv - \int v \, du
\]

The one-to-one correspondences are: \( u = f(x) \) and \( v = g(x) \). Then, \( du = \left(\frac{df}{dx}\right) \, dx \) and \( dv = \left(\frac{dg}{dx}\right) \, dx \).

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**Integration by Parts**

**Example # 1 / Part 1**

Evaluate the integral.

\[
\int x \cdot \cos(x) \, dx
\]

The one-to-one correspondences are: \( u = f(x) \) and \( v = g(x) \). Then, \( du = \left(\frac{df}{dx}\right) \, dx \) and \( dv = \left(\frac{dg}{dx}\right) \, dx \).

Let \( u = x \) and \( dv = \cos(x) \, dx \), then \( du = dx \) and \( v = \sin(x) \). Then plug those expressions into the Integration by Parts formula.

\[
\int x \cdot \cos(x) \, dx = x \cdot \sin(x) - \int \sin(x) \, dx
\]
**Integration by Parts**

**Example # 1 / Part 2**

**Recall**

\[\int x \cdot \cos(x) \, dx = x \cdot \sin(x) - \int \sin(x) \, dx\]

**But**

\[\int \sin(x) \, dx = -\cos(x) + C\]

**Therefore**

\[\int x \cdot \cos(x) \, dx = x \cdot \sin(x) + \cos(x) + C\]

---

**Integration by Parts**

**Evaluation of Definite Integrals**

**Use the Standard Form**

\[\int f(x) \frac{d}{dx} g(x) \, dx = f(x) \cdot g(x) - \int g(x) \frac{d}{dx} f(x) \, dx\]

**Apply the Fundamental Theorem of Calculus**

\[\int_{a}^{b} f(x) \frac{d}{dx} g(x) \, dx = (f(b) \cdot g(b) - f(a) \cdot g(a)) + \int_{a}^{b} g(x) \frac{d}{dx} f(x) \, dx\]
**Evaluation Of Definite Integrals**

**Example # 1 / Part 1**

Evaluate the integral.

\[ \int_{1}^{4} (\sqrt{t}) \cdot \ln(t) \, dt \]

Let \( u = \ln(t) \). That way we get \( du = \frac{1}{t} \cdot dt \) and \( \frac{1}{2} \) with \( dv = t^{\frac{1}{2}} \) that makes: \( v = \frac{2}{3} t^{\frac{3}{2}} \).

This eliminate the logarithm under the integral and reduce the integration problem to the application of the power law.

\[ \int_{1}^{4} (\sqrt{t}) \cdot \ln(t) \, dt = (\ln(4)) \cdot 2^{\frac{3}{2}} \cdot 4^{\frac{3}{2}} - (\ln(1)) \cdot 2^{\frac{3}{2}} \cdot 1^{\frac{3}{2}} - \int_{1}^{4} 2^{\frac{3}{2}} t^{\frac{3}{2}} \cdot \left(\frac{1}{t}\right) \, dt \]

**Evaluation of Definite Integrals**

**Example # 1 / Part 2**

\[ \int_{1}^{4} (\sqrt{t}) \cdot \ln(t) \, dt = \frac{16}{3} \cdot \ln(4) - \int_{1}^{4} 2^{\frac{3}{2}} t^{\frac{3}{2}} \cdot \left(\frac{1}{t}\right) \, dt \]

\[ \int_{1}^{4} 2^{\frac{3}{2}} t^{\frac{3}{2}} \cdot \left(\frac{1}{t}\right) \, dt = \frac{2}{3} \left[ \frac{3}{2} \cdot \ln(4) - \frac{3}{2}\right] = \frac{28}{9} \]

\[ \int_{1}^{4} (\sqrt{t}) \cdot \ln(t) \, dt = \frac{16}{3} \cdot \ln(4) - \frac{28}{9} = 4.282 \]
**Evaluation of Definite Integrals**

**Example # 1 / Part 3**

\[ f(t) := (\sqrt{t}) \cdot \ln(t) \quad F(t) := \int_{1}^{t} f(t) \, dt \]

![Graph of f(t) and F(t)]

**Evaluation of Definite Integrals**

**Example # 2 / Part 1**

First make a Substitution and then use Integration by Parts to evaluate the integral.

\[ \int_{1}^{4} x^2 e^x \, dx \]

Let \( t = \sqrt{x} \), then \( dt = \frac{1}{2\sqrt{x}} \, dx \). Solve for "dx" and find that: \( dx = 2\sqrt{x} \cdot dt = 2 \cdot t \cdot dt \)

\[ \int_{1}^{4} e^{\sqrt{x}} \, dx = \int_{\sqrt{1}}^{\sqrt{4}} 2 \cdot t \cdot e^t \, dt = 2 \int_{1}^{2} t \cdot e^t \, dt \]

The evaluation of this last definite integral can be done using *Integration by Parts*. 
Evaluation of Definite Integrals
Example # 2 / Part 2

\[ \int_{1}^{4} e^{\sqrt{x}} \, dx = 2 \cdot \int_{1}^{2} t \cdot e^{t} \, dt \]

Let \( u = t \), then \( du = dt \) and \( dv = e^{t} \, dt \) so that, \( v = e^{t} \).

Then,

\[ \int_{1}^{2} t \cdot e^{t} \, dt = \left[ (2) \cdot e^{2} \right] - \left[ (1) \cdot e^{1} \right] - \int_{1}^{2} e^{t} \, dt \]

\[ \int_{1}^{2} e^{t} \, dt = e^{2} - e^{1} \]

\[ \int_{1}^{2} t \cdot e^{t} \, dt = 2 \cdot e^{2} - e - e^{2} + e = e^{2} \]

\[ \int_{1}^{4} e^{\sqrt{x}} \, dx = 2 \cdot \int_{1}^{2} t \cdot e^{t} \, dt = 2 \cdot e^{2} = 14.778 \]

Evaluation of Definite Integrals
Example # 2 / Part 3

\[ f(x) := e^{\sqrt{x}} \quad F(x) := \int_{1}^{x} f(t) \, dt \]

\[ F(x) \quad f(x) \]

\[ \begin{array}{|c|c|}
\hline
x & F(x) \\
1 & 3 \\
2 & 6 \\
3 & 12 \\
4 & 15 \\
\hline
\end{array} \]

\[ x \]
Integration by Partial Fractions
Overview

The technique of Partial Fraction Expansions applies when the function you are trying to integrate is the RATIO OF TWO POLYNOMIALS!

$$\int \frac{P(x)}{Q(x)} \, dx$$

Integration by Partial Fractions
Preliminary Test

Expand $\frac{P(x)}{Q(x)}$ in Partial Fractions if the order of $Q(x)$ is strictly greater than the order of $P(x)$. Then find the antiderivatives of the Partial Fractions.

Otherwise, first divide $Q(x)$ into $P(x)$ to obtain a polynomial, $S(x)$, plus a "remainder". Expand the remainder and find its antiderivative and the antiderivative of $S(x)$. 
Evaluate the integral.

\[ \int \frac{x}{x - 5} \, dx \]

The integrand is a rational function or the ratio of two polynomials. This is a candidate for evaluation using Partial Fraction Expansion techniques.

Here, \( P(x) = x \) and \( Q(x) = x - 5 \). The order of the denominator is not greater than that of the numerator, so long division is a necessary preliminary.

\[ \int \frac{x}{x - 5} \, dx = \int 1 \, dx + \int \frac{5}{x - 5} \, dx \]

The last integral contains the remainder of the long division. Note that the remainder is also a ratio of two polynomials. But now the order of the numerator is less than the denominator. So, the remainder can be expanded directly.

But this integral is already so simple we can write its antiderivative by inspection.

\[ \int \frac{x}{x - 5} \, dx = x + 5 \ln(|x - 5|) + C \]
Integration by Partial Fractions

Expansion Process

• Factor the denominator.
• Examine the factored form.
• Categorize the factors.
• Equate the appropriate sum of generalized forms of Partial Fractions to the original proper fraction ratio of polynomials.
• Determine the values of their "unknown" coefficients so that the equation is true.
• Find the antiderivatives of all of the Partial Fractions and sum them algebraically.

Integration by Partial Fractions

“Distinct Linear Factors” Example / Part 1

Evaluate the following integral that has only distinct linear factors in the denominator.

\[ \int_{3}^{7} \frac{1}{(x + 1)(x - 2)} \, dx \]

Q(x) is already factored. Equate the sum of generalized forms of Partial Fractions appropriate for these factors that fall in the "distinct linear" category to the integrand.

\[ \frac{1}{(x + 1)(x - 2)} = \frac{A}{x + 1} + \frac{B}{x - 2} \]

Determine unknown coefficients: "A" & "B".

Multiply both sides of the equation by Q(x) and group coefficients of same powers of "x".

\[ (A + B) \cdot x^1 + (-2 \cdot A + B - 1) \cdot x^0 = 0 \]
**Integration by Partial Fractions**

**“Distinct Linear Factors” Example / Part 2**

\[(A + B) x^1 + (-2 A + B - 1) x^0 = 0\]

The only way the equation above can be true for all values of \(x\) is if the coefficient of each different power of \(x\) is separately equal to zero. So, we get two equations in the two “unknowns”: \(A\) and \(B\).

\[\begin{align*}
\int_{3}^{7} \frac{1}{(x + 1) (x - 2)} \, dx &= \frac{1}{3} \int_{3}^{7} \frac{1}{x + 1} \, dx - \frac{1}{3} \int_{3}^{7} \frac{1}{x - 2} \, dx \\
\end{align*}\]

These last two integrals have this antiderivative.

\[\int \frac{1}{x + a} \, dx = \ln |x + a|\]

I leave the rest of the details to you. The final value is 0.305.

**Integration by Partial Fractions**

**“Repeated Linear Factors” Example / Part 1**

Evaluate the integral that has repeated linear factors in the denominator

\[\int \frac{x^2}{(x - 3) (x + 2)^2} \, dx\]

\[\frac{x^2}{(x - 3)(x + 2)^2} = \frac{A}{x - 3} + \frac{B}{x + 2} + \frac{C}{(x + 2)^2}\]

Note the presence of two terms for the once repeated factor. Now do as before by multiplying thru by \(Q(x)\).

Group coefficients of same powers of \(x\).

\[(A + B - 1) x^2 + (2 A - B + C) x^1 + (4 A - 6 B - 3 C) x^0 = 0\]

The coefficients for each different power of \(x\) must separately be equal to zero. That gives us 3 equations for the 3 unknowns: \(A\), \(B\), and \(C\).
Integration by Partial Fractions
“Repeated Linear Factors” Example / Part 2

\[
\begin{align*}
A + B - 1 &= 0 \\
2A - B + C &= 0 \\
4A - 6B - 3C &= 0
\end{align*}
\]
Solve these equations "simultaneously".
\[
A = \frac{9}{25}, \quad B = \frac{16}{25}, \quad C = \frac{-4}{5}
\]

\[
\int \frac{x^2}{(x - 3)(x + 2)^2} \, dx = \frac{9}{25} \int \frac{1}{x - 3} \, dx + \frac{16}{25} \int \frac{1}{x + 2} \, dx + \frac{4}{5} \int \frac{1}{(x + 2)^2} \, dx
\]

\[
\int \frac{x^2}{(x - 3)(x + 2)^2} \, dx = \frac{9}{25} \ln |x - 3| + \frac{16}{25} \ln |x + 2| + \frac{4}{5} \frac{1}{(x + 2)^2} + C
\]

Integration by Partial Fractions
“Distinct Irreducible Quadratic Factors”

Evaluate the integral that has "distinct irreducible quadratic factors" in the denominator.

\[
\int \frac{2x + 3}{x(x^2 + 3)} \, dx
\]

Set it up this way and follow the same procedure demonstrated in the previous two examples.

\[
\frac{2x + 3}{x(x^2 + 3)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 3}
\]
Integration by Partial Fractions  
“Repeated Irreducible Quadratic Factors”

Evaluate the integral that has "repeated irreducible quadratic factors" in the denominator

\[
\int \frac{x^4 + 1}{x(x^2 + 1)^2} \, dx
\]

Set it up this way and follow the same procedure demonstrated in the earlier, worked-out two examples.

\[
\frac{x^4 + 1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{B \cdot x + C}{x^2 + 1} + \frac{D \cdot x + E}{(x^2 + 1)^2}
\]

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