tDEC-112 Lecture # 8

Applications Overview

• Calculus is used in many disciplines.
• Physics & Engineering are the principal ones.
• “Work”, Hydrostatic Pressure & Force, and Moments & Centers of Mass are the focus here.
A bucket that weighs 4 lb and a rope of negligible weight are used to draw water from a well that is 80 ft deep. The bucket starts with 40 lb of water and is pulled up at a rate of 2 ft/sec, but water leaks out of the bucket at a rate of 0.2 lb/sec. Find the work done in pulling the bucket to the top of the well.

\[ dW = F \cdot dx = (4 + F_{\text{water}}) \cdot dx \]

After rising a distance, \( x \), the bucket has been losing water for \( t \) seconds.

\[ t = \frac{x}{2} \]

The weight of water remaining after \( t \) seconds is the initial 40 lb less the amount that leaked.

\[ F_{\text{water}} = 40 - \left( \frac{0.2}{2} \right) x = 40 - 0.1 \cdot x \]

"dW" can now be expressed explicitly in terms of "x" alone.

\[ dW = \left[ 4 + 40 - 0.1 \cdot x \right] \cdot dx \]

The total work, \( W \), is the sum of all of the infinitesimal contributions.

\[ W = \int_{0}^{80} \left[ 4 + 40 - 0.1 \cdot x \right] \, dx = 3.2 \times 10^3 \, \text{ft-lb} \]
“Work”

Example # 2 / Part 1

A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high and the depth of water is 4 ft. How much work is required to pump all of the water out over the side?

The work, "dW", required to pump out the thin, circular, slice of water is the product of the weight, "dFwater", of that slice and the distance, "x", that it must be moved.

\[
dW = x \cdot dF_{\text{water}}
\]

Example # 2 / Part 2

The weight of the slice is the product of its volume, "dV", and its density, "\( \rho = 62.4 \text{ lb/ft}^3 \)".

\[
dF_{\text{water}} = 62.4 \cdot dV
\]

The volume, "dV", is the product of the circular area, "A = \pi (12)^2", and the thickness, "dx".

\[
dV = 144 \pi \cdot dx
\]

The weight of the slice of water and finally the work to remove it can be expressed in terms of "x" alone.

\[
dF_{\text{water}} = (62.4 \cdot 144 \pi \cdot dx) = 2.823 \times 10^4 \cdot dx
\]

\[
dW = 2.823 \times 10^4 \cdot x \cdot dx
\]

Add all of these infinitesimal amounts of work to find the total work, "W", from the bottom of the tank to the 4 ft water level.

\[
W = \int_1^5 2.823 \times 10^4 \cdot x \, dx = 3.388 \times 10^5 \text{ ft-lb}
\]
A vertical dam has a trapezoidal gate. (The gate's left side has the same dimensions as its right side, so it is symmetric.). Find the net force exerted on the gate by the water.

A distance, "x", from the bottom of the dam is a tiny, horizontal rectangular piece of the gate of thickness, "dx". Because it is infinitesimally small in the vertical direction, the pressure acting on it, all along its length, is constant.

Express the force, "dF", that this rectangle is subjected to in terms of the pressure, "P(x)", that acts across its entire length, and its area, "dA".

\[ dF = P(x) \cdot dA \]

The area, "dA", of the rectangle is the product of its length, "L(x)" and its width, "dx".

\[ dA = L(x) \cdot dx \]

Use similar triangles to express "L(x)" in terms of "x".

\[ L(x) = 4 - \frac{x}{2} \]

"dA" can now be expressed in terms of "x" alone.

\[ dA = \left(4 - \frac{x}{2}\right) dx \]
This leaves only the pressure, "P(x)", to be determined so that "dF" can be expressed explicitly in terms of "x".

\[ dF = P(x) \left( 4 - \frac{x}{2} \right) \, dx \]

The pressure, "P(x)", acting on the tiny rectangle is equal to the product of the height, "h(x)", of the water above it and the density:

\[ P(x) = \rho \cdot g \cdot h(x) = 9800 \cdot h(x) \]

The height, "h(x)", can be found directly from the diagram.

\[ h(x) = 10 - x \]

**Hydrostatic Pressure & Force**

**Example # 1 / Part 4**

Make the appropriate substitutions and find "dF" explicitly in terms of "x".

\[ dF = 9800 \cdot (10 - x) \left( 4 - \frac{x}{2} \right) \, dx \]

Add all of the infinitesimal forces to find the TOTAL FORCE, "F", acting on the gate.

\[ F = \int_{0}^{4} 9800 \cdot (10 - x) \left( 4 - \frac{x}{2} \right) \, dx = 9.669 \times 10^5 \, \text{n} \]
**Hydrostatic Pressure & Force**

**Example # 2 / Part 1**

The end of a tank containing water is vertical and has a circular shape. Determine the hydrostatic force pushing against the end of the tank.

The thin horizontal, rectangular slice of the tank is subjected to the same pressure, "P(x)”, everywhere on the rectangle of area, "dA”. The product of the pressure and the area that that pressure acts on produces a force, "dF”.

\[ dF = P(x) \cdot dA \]

The area, "dA”, of the rectangle is the product of its length, "L(x)”, and its width, "dx”.

\[ dA \approx L(x) \cdot dx \]

**Hydrostatic Pressure & Force**

**Example # 2 / Part 2**

The length, "L(x)”, is determined by the Pythagorean theorem where the hypotenuse is the radius, "10 m” and "L(x)" is twice the base.

\[ L(x) = 2 \sqrt{10^2 - x^2} \]

"dA” can now be expressed in terms of "x” alone.

\[ dA \approx 2 \sqrt{10^2 - x^2} \cdot dx \]

This leaves only the pressure, "P(x)”, to be determined so that "dF” can be expressed explicitly in terms of "x”.

\[ dF = P(x) \cdot (2 \sqrt{10^2 - x^2} \cdot dx) \]

The pressure, "P(x)”, is equal to the product of the height, "h(x)”, of the water above it and the density, "\( \rho \cdot g \frac{m}{m^3} \)".

\[ P(x) = 9800 \cdot \frac{n}{m^3} \cdot h(x) \]
**Hydrostatic Pressure & Force**

**Example # 2 / Part 3**

Finally, the height, "h(x)" can be found directly from the diagram.

\[ h(x) = x - 5 \]

Make the appropriate substitutions and find "dF" explicitly in terms of "x".

\[ dF = 9800 \cdot (x - 5) \cdot 2 \sqrt{10^2 - x^2} \ dx \]

Add all of the infinitesimal forces to find the total force, "F", acting on the end of the tank.

\[ F = \int_{5}^{10} 9800 \cdot (x - 5) \cdot 2 \sqrt{10^2 - x^2} \ dx \approx 1.234 \times 10^6 \text{ N} \]

**Moments & Centers-of-Mass**

**Example # 1 / Part 1**

Calculate the moments: "M_x" and "M_y" and the coordinates, "(x_c, y_c)" of the center of mass of a lamina with density, "\( \rho = 1 \)" and with the shape shown in the diagram.

A lamina is a flat plate of uniform thickness. If its density is also the same throughout, then the coordinates of the centroid, or "balance-point", are determined solely by its shape. Guessing from the diagram, \((0, 1/3)\), looks close to the balance-point, which can be determined exactly.

First, find the area, "A", of the region. "A" in this case is the area of an isosceles triangle.

\[ A = \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 2 \cdot 2 = 2 \]
Moments & Centers-of-Mass
Example # 1 / Part 2

Second, find the infinitesimal moments:
"dM_y" and "dM_x" about the "x" and "y" axes of a vertical rectangle of width, "dx", and height, "h(x)". 

"dM_y" equals the product of the area, "dA", of the vertical rectangle and the x-coordinate location of the rectangle.

\[ dM_y = x \cdot h(x) \cdot dx \]

"dM_x" equals the product of the area, "dA", of the vertical rectangle and 1/2 of the y-coordinate location of the rectangle (because the centroid of the little rectangle is located at the rectangle’s center).

\[ dM_x = \frac{h(x)}{2} \cdot h(x) \cdot dx = \frac{(h(x))^2}{2} \cdot dx \]

"h(x)" is the y-coordinate of the lamina.

\[ h(x) = 2 \cdot x + 2 \quad -1 \leq x \leq 0 \]
\[ h(x) = -2 \cdot x + 2 \quad 0 \leq x \leq 1 \]

Moments & Centers-of-Mass
Example # 1 / Part 3

The total moment, "M_y", is the sum of infinitesimal contributions which add to zero.

\[ M_y = \int_{-1}^{1} \rho x \cdot h(x) \, dx = 0 \]

The total moment, "M_x", is the sum of infinitesimal contributions that add to this.

\[ M_x = \int_{-1}^{1} \rho x \cdot h(x) \, dx = 2 \int_{0}^{1} \frac{(2 - 2x)^2}{2} \, dx = \frac{4}{3} \]

The centroid coordinates are these.

\[ x_c = \frac{M_y}{A} = \frac{0}{2} = 0 \quad y_c = \frac{M_x}{A} = \frac{4}{3 \cdot 2} = \frac{2}{3} \]
\[ (x_c, y_c) = \left( 0, \frac{2}{3} \right) \]
Moments & Centers-of-Mass

Example # 2 / Part 1

A "slab" of uniform thickness and density, \( \rho \), is shown in a "top-view" in the figure below. The area that that top-view occupies is bounded by the functions: \( \frac{1}{\sqrt{x}} \), \( -\frac{1}{\sqrt{x}} \), \( x = 0 \), and \( x = 2 \). Find its moments about the x & y axes and its center-of-mass or "centroid".

\[ \int_{0}^{2} \rho \cdot \frac{1}{\sqrt{x}} \cdot (\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}}) \cdot dx = \frac{\rho}{2} \]

Find the infinitesimal moments: \( dM_x \) and \( dM_y \) about the "x" and "y" axes of a vertical rectangle of width, "dx", and height, "h(x)".

\[ W = \rho \cdot \frac{1}{\sqrt{x}} \cdot \lim_{a \to 0^+} \frac{1}{\sqrt{x}} \cdot h(x) \cdot dx = \frac{\rho}{2} \cdot \frac{1}{\sqrt{x}} \cdot h(x) \cdot dx \]

Find the weight, "W", of the slab.

Moments & Centers-of-Mass

Example # 2 / Part 2

"dM_x" equals the product of the area, "dA", of the vertical rectangle, its density, \( \rho \) and the x-coordinate location of the rectangle.

\[ dM_x = \rho \cdot h(x) \cdot dx \]

The height, "h(x)" of the vertical rectangle equals the difference between the upper and lower boundaries of the lamina.

\[ h(x) = \frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x}} = \frac{2}{\sqrt{x}} \]

"dM_y" equals the product of the area, "dA", of the vertical rectangle, its density, \( \rho \) and the y-coordinate location of its center.

\[ dM_y = \rho \cdot \left( \frac{h(x)}{2} \right) \cdot dx \]
Moments & Centers-of-Mass

Example # 2 / Part 3

The total moment, \( M_x \), is the sum of its infinitesimal parts.

\[
M_x = \int_{-a}^{a} \rho \left( \frac{1}{2} \right) \frac{2}{\sqrt{x}} \, dx = 0
\]

The total moment, \( M_y \), is the sum of all of the infinitesimals.

\[
M_y = \int_{-a}^{a} \rho x \left( \frac{2}{\sqrt{x}} \right) \, dx = 3.771 \rho
\]

These are the coordinates of the centroid.

\[
\begin{align*}
x_c &= \frac{M_y}{A} = \frac{3.771 \rho}{5.657 \rho} = 0.667 \\
y_c &= \frac{M_x}{A} = \frac{0}{5.657 \rho} = 0
\end{align*}
\]

\( (x_c, y_c) = (0.667, 0) \)

Black Slide