tDEC-112 Lecture # 9

Exponential Growth & Decay
Overview

- A quantity that changes with time at a constant relative rate either grows or decays exponentially.
- A positive constant relative rate produces growth.
- A negative constant relative rate produces decay.
Solving the Differential Equation

Part 1

The relative rate of the quantity, "Q(t)"., is a constant, "k".

\[ \frac{d}{dt} \frac{Q}{Q} = k \]

Solve this equation for "Q(t)" to determine its time-dependence.

Multiply both sides by "Q(t)".

\[ \frac{d}{dt} Q = k \cdot Q \]

Separate the variables: "Q(t)" and "t" by multiplying thru by "dt" and dividing by "Q(t)".

\[ \frac{dQ}{Q} = k \cdot dt \]

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Solving the Differential Equation

Part 2

Integrate both sides, using "dummy" variables in the integrands.

\[ \int_{Q(0)}^{Q(t)} \frac{1}{x} \, dx = \int_{0}^{t} k \, ds \]

Apply the Evaluation Theorem

\[ \ln \left( \frac{Q(t)}{Q(0)} \right) = k \cdot t \]

Exponentiate both sides of the equation.

\[ \frac{Q(t)}{Q(0)} = e^{k \cdot t} \]

The quantity can never be negative.

\[ Q(t) = Q(0) \cdot e^{k \cdot t} \]

This equation says that the initial quantity grows exponentially if k > 0 and decays exponentially if k < 0. (If k = 0, the Q(t) = Q(0)).
**Exponential Growth**

**Example #1 / Part 1**

One cell of a bacterium divides into two cells every 20 minutes. The initial population is '60" cells. Demonstrate how to answer the typical questions asked in connection with examples like this one of "Exponential Growth"

The relative growth rate is "k" and if "P(t)" is the population, then "k" can be determined from the information that was provided using the solution to the differential equation.

\[ P(t) = P(0) \cdot e^{k \cdot t} \]

The population doubles every 20 minutes, or 1/3 hours. With an initial population of "60", after, 1/3 hours later it will double to "120"

\[ P\left(\frac{1}{3}\right) = 120 = 60 \cdot e^{k \cdot \frac{1}{3}} \]

Solve for "k".

\[ k = 3 \cdot \ln(2) \]

**Exponential Growth**

**Example #1 / Part 2**

The population after "t" hours can now be explicitly expressed.

\[ P(t) = 60 \cdot e^{3 \cdot \ln(2) \cdot t} = 60 \left( e^{\ln(2^3)} \right)^t = 60 \cdot (8)^t \]

The population after 8 hours is determined by substituting "8" for "t".

\[ P(8) = 60 \cdot (8)^8 = 1.007 \times 10^9 \text{ cells} \]

The time, "t₀", required for the initial population of 60 to grow to a population 20000 is found by solving this equation for "t₀".

\[ \frac{20000}{60} = (8)^t₀ \]

\[ \ln\left(\frac{20000}{60}\right) = t₀ \cdot \ln(8) \]

\[ t₀ = \frac{\ln(20000) - \ln(60)}{\ln(8)} = 2.794 \text{ hours} \]
Exponential Decay
Example #1 / Part 1

After 3 days, a sample of radon-222 decayed to 58% of its original amount. What is the half-life of radon-222 and how long would it take the sample to decay to 10% of its original amount?

The isotope has a constant relative decay rate and if "m(t)" is the mass, then "k" can be determined from the information that was provided using the solution to the differential equation.

\[ m(t) = m(0) e^{-kt} \]

The half-life, "\( \tau \)" is the time it takes for 50% of the original mass, "m(0)", of that radioactive isotope to lose its "extra" neutrons. That time is a constant.

In 3 days 58% of the original isotope remains.

\[ \frac{m(3)}{m(0)} = 0.58 = e^{k(3)} \]

Solve for "k".

\[ k = \frac{1}{3} \ln(0.58) = \ln\left(\sqrt[3]{0.58}\right) \]

Exponential Decay
Example #1 / Part 2

The half-life is the solution, "\( \tau \)" to this equation.

\[ \frac{m(\tau)}{m(0)} = 0.5 = e^{\left(\ln(0.58)\right) \tau} \]

\[ \tau = \frac{3 \ln(0.58)}{\ln(0.58)} = 3.82 \text{ days} \]

The fractional mass remaining after "t" days can be explicitly expressed because 'k' is known.

\[ \frac{m(t)}{m(0)} = e^{\left(\frac{\ln(0.58)}{3}\right) t} = e^{-0.182 \cdot t} \]

The time, "\( t_0 \)" for 90% of the isotope to decay is the solution to this equation.

\[ 0.1 = e^{-0.182 \cdot t_0} \]

\[ t_0 = \left(\frac{1}{0.182}\right) \ln(0.1) = 12.65 \text{ days} \]

An alternative, but equivalent expression for this process of radioactive decay is this.

\[ m(t) = m(0) \cdot 2^{-\frac{t}{3.82}} \]
Inhibited Growth Model
Overview

• Most growth is inhibited.
• As the population rises, opportunity for further increases falls.
• The first order logistic differential equation is the simplest that adequately accounts for this more realistic behavior.

Solving the Logistic Equation
Part 1

The relative rate of the "infected" population, "y(t)", varies directly with those who have not yet contracted the disease. If "L" is the disease "carrying capacity" of the total population, then the difference: " ( L - y(t) ) " is the number of additional hosts that the disease can potentially infect and remain infected.

\[
\frac{d}{dt} \left( \frac{y}{y} \right) = k \cdot \left( 1 - \frac{y}{L} \right)
\]

\[
\frac{dy}{y \cdot \left( 1 - \frac{y}{L} \right)} = k \cdot dt
\]

Integrate both sides using "dummy" variables in the integrands.
Solving the Logistic Equation
Part 2

\[ \frac{1}{y(0)} \int_{y(0)}^{y(t)} \frac{1}{x(1 - \frac{x}{L})} \, dx = \int_{0}^{t} k \, dt = k \cdot t \]

Partial Fractions apply.

\[ \frac{L}{x(L-x)} = \frac{1}{x} - \frac{1}{L-x} \]

\[ k \cdot t = 1 \cdot \int_{y(0)}^{y(t)} \frac{1}{x} \, dx + 1 \cdot \int_{y(0)}^{y(t)} \frac{1}{L-x} \, dx \]

Exponentiate both sides of the equation.

\[ e^{k \cdot t} = \frac{y(t) \cdot (L-y(0))}{y(0) \cdot (L-y(t))} \]

Solving the Logistic Equation
Part 3

Solve for \( y(t) \).

\[ y(t) = \frac{y(0)}{1 - \frac{y(0)}{L}} \cdot e^{-(k \cdot t)} + \frac{y(0)}{L} \]

- \( y(0) > L \): Carrying Capacity Exceeded
- \( y(0) < L \): Capacity Not Exceeded
- \( L \to \infty \): Uninhibited Behavior
Polar Coordinates
Overview

- The Polar Coordinate System is a popular alternative to the Cartesian Coordinate System.
- Point locations in the plane can be specified using either system.
- Some problems, particularly those evidencing circular symmetry, are best expressed using Polar Coordinates.

Polar Coordinates
Definition

Polar - to - Cartesian

\[ x = r \cdot \cos(\theta) \]
\[ y = r \cdot \sin(\theta) \]

Cartesian - to - Polar

\[ r^2 = x^2 + y^2 \]
\[ \tan(\theta) = \frac{y}{x} \]
**Polar Coordinates**

**Example # 1**

Find a polar equation for the curve represented by the given Cartesian equation.

\[ y = 2x - 1 \]

Use the System-to-System Conversion Equations.

\[ r \cdot \sin(\theta) = 2 \cdot r \cdot \cos(\theta) - 1 \]

Solve for "r" in terms of "\( \theta \)".

\[ r(\theta) := \frac{1}{2 \cdot \cos(\theta) - \sin(\theta)} \]

Construct a Polar Plot.

**Polar Coordinates**

**Example # 2**

Sketch the curve with the given Polar Equation.

\[ r(\theta) := 2 + \cos(\theta) \]

Polar Coordinate Plot
Polar Coordinates
Example # 3 / Part 1

Find the points on the given curve where the
tangent line is horizontal or vertical.

\[ r^2 = \sin(2 \theta) \]

Vertical tangents imply that \( \frac{dy}{d\theta} = 0 \).

Horizontal tangents imply that \( \frac{dx}{d\theta} = 0 \).

Thus, find \( \frac{dy}{d\theta} \) and \( \frac{dx}{d\theta} \).

Differentiate implicitly to find \( \frac{dr}{d\theta} \).

\[
\frac{dr}{d\theta} = \frac{d}{d\theta} \left( \sin(2\theta) \right) = 2 \cdot \frac{d}{d\theta} r = 2 \cdot \cos(2\theta)
\]

\[
\frac{dr}{d\theta} = \frac{\cos(2\theta)}{r}
\]

Polar Coordinates
Example # 3 / Part 2

\[
y = r \sin(\theta) \\
\frac{dy}{d\theta} = \sin(\theta) \frac{dr}{d\theta} + r \cos(\theta)
\]

\[
\frac{dr}{d\theta} = \frac{\cos(2\theta)}{r}
\]

\[
\frac{d}{d\theta} y = \sin(\theta) \left( \frac{\cos(2\theta)}{r} \right) + r \cos(\theta) = \frac{1}{r} \sin(3\theta)
\]

\[
\frac{d}{d\theta} y = \frac{1}{r} \sin(3\theta)
\]

\[
x = r \cos(\theta) \\
\frac{dx}{d\theta} = -r \sin(\theta) + \cos(\theta) \frac{dr}{d\theta}
\]

\[
\frac{dr}{d\theta} = \frac{\cos(2\theta)}{r}
\]

\[
\frac{d}{d\theta} x = -r \sin(\theta) + \cos(\theta) \left( \frac{\cos(2\theta)}{r} \right) = \frac{\cos(3\theta)}{r}
\]

\[
\frac{dx}{d\theta} = \frac{1}{r} \cos(3\theta)
\]
Polar Coordinates
Example # 3 / Part 3

Horizontal tangents, occur where \( \sin(3\theta) = 0 \) and \( \sin(2\theta) \geq 0 \).

\[ \theta = 0, \frac{1}{3}\pi, \pi, \frac{4}{3}\pi \]  

The Polar Points for these values of "\( \theta \)" are:

\[ (0, 0), (0.931, \frac{\pi}{3}), (0, \pi), (0.931, \frac{4\pi}{3}) \]

Vertical tangents, occur where \( \cos(3\theta) = 0 \) and \( \sin(2\theta) \geq 0 \).

\[ \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{7}{6}\pi, \frac{3}{2}\pi \]  

The Polar Points for these values of "\( \theta \)" are:

\[ (0.931, \frac{\pi}{6}), (0, \frac{\pi}{2}), (0.931, \frac{7\pi}{6}), (0, \frac{3\pi}{2}) \]

Arc Length for Polar Curves
Overview

If no segment of the polar curve is traced more than once over the curve and if its derivative is continuous from the smallest to the largest value of the polar angle, then this is the arc length.

\[ L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta \]
Arc Length for Polar Curves
Example # 1 / Part 1

Find the total arc length of the spiral.

\[ r(\theta) := e^{-\frac{\theta}{8}} \quad 0 \leq \theta \leq \infty \]

Use the template.

\[ L = \int_{\alpha}^{\beta} \sqrt{\left( r'(\theta) \right)^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta \]

Perform the needed calculations and make the appropriate identifications.

\[ \left( \frac{dr}{d\theta} \right)^2 = \frac{e^{-\frac{\theta}{4}}}{64} \quad r^2 = e^{-\frac{\theta}{4}} \]

\[ \alpha = 0 \quad \beta = \infty \]

\[ L = \int_{0}^{\infty} \sqrt{\frac{65}{64}e^{-\frac{\theta}{4}}} \, d\theta = \frac{65}{64} \int_{0}^{\infty} \frac{-\theta}{4} e^{-\frac{\theta}{4}} \, d\theta \]

Make the Improper integral proper.

\[ L = 1.01 \cdot \lim_{\theta \to \infty} \int_{0}^{\theta} \frac{-\theta}{4} e^{-\frac{\theta}{4}} \, d\theta \]

Evaluate the definite integral.

\[ 4 \cdot \lim_{t \to \infty} \int_{0}^{\frac{t}{4}} e^{-u} \, du = 4 \cdot \lim_{t \to \infty} \left( 1 - e^{-t} \right) \]

Find the limit.

\[ \lim_{t \to \infty} \left( 1 - e^{-t} \right) = 1 \]

Substitute the value of the limit to determine "L".

\[ L = 1.008 \cdot (4) \cdot (1) = 4.032 \]

Arc Length for Polar Curves
Example # 1 / Part 2

Plot the curve.

SPIRAL

90

180

270
Black Slide