Part I. The problems on this part are of multiple-choice type. Circle the correct answer. (5 pts each)

1. Describe the surface in $\mathbb{R}^3$ that is represented by the equation $x^2 + z^2 = 9$.
   
   A) horizontal plane  
   B) sphere  
   C) cylinder around the x-axis  
   D) vertical plane  
   E) cone  
   F) cylinder around the y-axis

**Solution.** The equation $x^2 + z^2 = 9$ places no restriction on the value of $y$. In the $xz$-plane it represents a circle of radius 3 with center at the origin. In $\mathbb{R}^3$ it represents a cylinder of radius 3 around the y-axis. The correct answer is F.

2. Find the work done by a constant force $\mathbf{F} = i + 2j + 2k$ (magnitude = 3 N) in moving an object along the z-axis from the origin to the point (0,0,5) (distance in meters).
   
   A) 5 J  
   B) 6 J  
   C) 9 J  
   D) 10 J  
   E) 12 J  
   F) 15 J

**Solution.** The work is done by the scalar component of the force in the direction of the motion, i.e. in the direction of the displacement vector $\mathbf{d} = 5k$. Thus $W = (2)|\mathbf{d}| = (2)(5) = 10$, or (equivalently) $W = \mathbf{F} \cdot \mathbf{d} = (1)(0) + (2)(0) + (2)(5) = 10$. The correct answer is D.

3. The domain of the function $f(x,y) = \frac{\ln(x^2+y^2)}{\sqrt{x}}$ is the set of points (x,y) for which:
   
   A) $x > 0$  
   B) $y > 0$  
   C) $x > 0$ and $y > 0$  
   D) $x \geq 0$  
   E) $y \geq 0$  
   F) $x < 0$ and $y < 0$

**Solution.** We must have $x > 0$ because of $\sqrt{x}$ appearing in the denominator. This ensures that $\ln(x^2+y^2)$ is defined (since $x^2+y^2 > 0$), so there are no further restrictions. The correct answer is A.

4. Find the limit $\lim_{(x,y) \to (0,0)} \frac{x-y}{x+y}$ if it exists.
   
   A) 0  
   B) 1  
   C) $-1$  
   D) $\infty$  
   E) $-\infty$  
   F) Does not exist

**Solution.** Along the x-axis ($y = 0$) we have $\lim_{(x,y) \to (0,0)} \frac{x-y}{x+y} = \lim_{x \to 0} \frac{x}{x} = 1$, and along the y-axis ($x = 0$) we have $\lim_{(x,y) \to (0,0)} \frac{x-y}{x+y} = \lim_{y \to 0} \frac{-y}{y} = -1$. Since these directional limits have different values, the two-dimensional limit $\lim_{(x,y) \to (0,0)} \frac{x-y}{x+y}$ does not exist. The correct answer is F.
5. The various parts of this problem all refer to the points K(1,1,1), L(1,2,–1), and M(3,0,2) in $\mathbb{R}^3$.

(a) (8 pts) Find (to the nearest degree) the angle between the vectors $\overrightarrow{KL}$ and $\overrightarrow{KM}$.

**Solution.** $\overrightarrow{KL} = \langle 0, 1, -2 \rangle$ and $\overrightarrow{KM} = \langle 2, -1, 1 \rangle$; thus the angle between these vectors is given by

$$
\cos \theta = \frac{\overrightarrow{KL} \cdot \overrightarrow{KM}}{|\overrightarrow{KL}| |\overrightarrow{KM}|} = \frac{(0)(2) + (1)(-1) + (-2)(1)}{\sqrt{(0)^2 + (1)^2 + (-2)^2} \sqrt{(2)^2 + (-1)^2 + (1)^2}} = -\frac{3}{\sqrt{5} \sqrt{6}}
$$

$$
\theta = \cos^{-1} \left( -\frac{3}{\sqrt{30}} \right) \approx \cos^{-1} (-0.5477) \approx 123^\circ
$$

(b) (8 pts) Find a unit vector that is perpendicular to the plane determined by K, L, and M.

**Solution.** We obtain a vector perpendicular to the plane by taking the cross product of $\overrightarrow{KL}$ and $\overrightarrow{KM}$.

$$
\overrightarrow{KL} \times \overrightarrow{KM} = \begin{vmatrix} i & j & k \\ 0 & 1 & -2 \\ 2 & -1 & 1 \end{vmatrix} = (1 - 2)i - (0 + 4)j + (0 - 2)k = -i - 4j - 2k
$$

There are two possibilities for a unit vector perpendicular to the plane:

$$
\mathbf{n} = \pm \frac{\overrightarrow{KL} \times \overrightarrow{KM}}{|\overrightarrow{KL} \times \overrightarrow{KM}|} = \pm \frac{-i - 4j - 2k}{\sqrt{21}}
$$

(c) (6 pts) Find the area of the triangle having K, L, and M as its vertices.

**Solution.** The area of the triangle is one-half the length of the cross product $\overrightarrow{KL} \times \overrightarrow{KM}$.

$$
A = \frac{1}{2} |\overrightarrow{KL} \times \overrightarrow{KM}| = \frac{1}{2} \sqrt{21} \approx 2.29
$$
6. This problem is concerned with the line \( L \) that passes through the point \( P(1,-1,2) \) and is parallel to the vector \( \mathbf{u} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k} \).

(a) (6 pts) Find parametric equations for the line \( L \).

**Solution.** Using \( P(1,-1,2) \) as the base point, and \( \mathbf{u} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k} \) as the direction, the line \( L \) has parametric equations

\[
\begin{align*}
x &= 1 + 3t \\
y &= -1 - t \\
z &= 2 - 2t
\end{align*}
\]

(b) (6 pts) Find the coordinates of the point where the line \( L \) intersects the plane \( x + z = 1 \).

**Solution.** The value of \( t \) corresponding to \( x + z = 1 \) is

\[
(1 + 3t) + (2 - 2t) = 1 \quad \text{and} \quad (3t - 2t) + (1 + 2) = 1 \quad \Rightarrow \quad t = -2
\]

and, substituting this \( (t = -2) \) back into the parametric equations, we obtain \( x = -5, y = 1, z = 6 \). Thus the point of intersection is \( (-5,1,6) \).

(c) (6 pts) Find an equation for the plane that is perpendicular to \( L \) and passes through \( P(1,-1,2) \).

**Solution.** Using \( P(1,-1,2) \) as base point, and \( \mathbf{u} = 3\mathbf{i} - \mathbf{j} - 2\mathbf{k} \) as the normal direction, the equation of the plane is given by

\[
3(x - 1) - (y + 1) - 2(z - 2) = 0 \quad \text{or} \quad 3x - y - 2z = 0
\]

7. (6 pts) Use the axes provided to sketch the graph of the surface with equation \( z = \sqrt{x^2 + y^2} \).

What type (name) of surface is it?

**Solution.** Since \( z \geq 0 \), the surface lies above the xy-plane. The trace of the surface in a horizontal plane \( z = k \) \((k > 0)\) is the circle \( x^2 + y^2 = k^2 \).

The surface is a cone with vertex at the origin and central axis along the z-axis.
8. This problem is concerned with the space curve C with vector equation \( r(t) = \langle 2 \cos(t), 2 \sin(t), t \rangle \).

(a) (6 pts) Use the axes provided to describe and sketch the curve C. What type (name) of curve is it?

**Solution.** Note that all of the points on the curve satisfy
\[
x^2 + y^2 = 4 \cos^2 t + 4 \sin^2 t = 4
\]
Thus the curve lies on the surface of the cylinder \( x^2 + y^2 = 4 \), with the \( z \)-coordinate increasing as \( t \) increases (\( z = t \)).

The curve is a *helix*, starting at the point \((2,0,0)\) (when \( t = 0 \)) and climbing up the outside of the cylinder \( x^2 + y^2 = 4 \).

(b) (6 pts) Find a unit vector which is tangent to C at the point corresponding to \( t = \frac{\pi}{3} \).

**Solution.** The unit tangent vector is \( T(t) = \frac{v(t)}{|v(t)|} \) where \( v(t) = r'(t) = \langle -2 \sin(t), 2 \cos(t), 1 \rangle \), and
\[
|v(t)| = \sqrt{4 \sin^2 t + 4 \cos^2 t + 1} = \sqrt{5}.
\]
Thus, for \( t = \frac{\pi}{3} \) we have
\[
T\left(\frac{\pi}{3}\right) = \frac{\langle -2(\frac{\sqrt{3}}{2}), 2(\frac{1}{2}), 1 \rangle}{\sqrt{5}} = \frac{\langle -\sqrt{3}, 1, 1 \rangle}{\sqrt{5}}
\]

(c) (6 pts) Find the arc length of the segment of the curve C corresponding to \( 0 \leq t \leq \pi \).

**Solution.** We get arc length by integrating \( \frac{ds}{dt} = |v(t)| = \sqrt{5} \); thus
\[
L = \int_0^\pi \sqrt{5} \, dt = \sqrt{5} \pi \approx 7.02
\]
9. (8 pts) Suppose the function $T(x,y) = y - x^2$ represents the temperature (in degrees centigrade) at the point $(x,y)$ in a flat metal plate. Use the axes provided to sketch a contour map for $T$ showing the isotherms (level curves) corresponding to $T = 0$, $T = -1$, and $T = 2$.

**Solution.** The level curve corresponding to $T = 0$ is

$$y - x^2 = 0 \quad \text{or} \quad y = x^2$$

Similarly, the level curves corresponding to $T = -1$ and $T = 2$ are

$$y = x^2 - 1 \quad \text{and} \quad y = x^2 + 2$$

10. (8 pts) Find the velocity vector and position vectors of a particle, given that the acceleration vector is $a(t) = \mathbf{i} - \mathbf{k}$ and the initial velocity and position are $v(0) = \mathbf{i} + \mathbf{j}$ and $r(0) = \mathbf{0}$.

**Solution.** Velocity is the antiderivative of acceleration:

$$v(t) = t\mathbf{i} - tk + C_1 \quad \text{where (setting } t = 0) \quad C_1 = v(0) = \mathbf{i} + \mathbf{j}$$

$$v(t) = (t + 1)\mathbf{i} + \mathbf{j} - tk$$

Position is the antiderivative of velocity:

$$r(t) = (\frac{1}{2} t^2 + t)\mathbf{i} + tj - \frac{1}{2} t^2\mathbf{k} + C_2 \quad \text{where} \quad C_2 = r(0) = \mathbf{0}$$

$$r(t) = (\frac{1}{2} t^2 + t)\mathbf{i} + tj - \frac{1}{2} t^2\mathbf{k}$$