Chapter 12 Calculus based optimization -- finding minima and maxima

Section 12.1 Finding minima and maxima by using Maple as a calculus calculator

A standard topic in elementary calculus is how to find the minimum or maximum of a continuous function in an interval (sometimes referred to as the *local extrema of the function*) through the use of derivatives and a little algebra. The function that you are doing this to is sometimes referred to as the *objective function*. Maple can be used to do the same calculations as you would do by hand -- to find the extrema of the objective function, take its derivative with respect to x (using *diff*), and then find where the derivative is equal to zero using *solve* or *fsolve*. The advantages of doing it with a system such as Maple are the same as with other calculations: a) it is easier to do correctly if the formula is involved or if you don't remember all the math, b) you can easily organize the information about the problem and its solution using Maple documents so that it can be easier to recall and presented, c) the speed of re-execution makes it easy to handle a bundle of similar extrema problems.

### Example 12.1.1 Finding the minimum and maximum using textually-specified commands

From Anton, Calculus 8th ed. p. 307 problem 13 (modified)

Find the absolute maximum and minimum values of

\[ f(x) = \frac{2 - 2 \sin x}{x} \] for \( x \) in \( \pi \frac{4}{4} .. \pi \)

**Solution**

\[ f := (x) \rightarrow 2 - 2 \cdot \sin(x) \] for \( x \) in \( \pi \frac{4}{4} .. \pi \)  

\[ x \rightarrow 2 - 2 \sin(x) \]  

\[ \text{deriv := } \text{diff}(f(x), x) \]  

\[ -2 \cos(x) \]  

\[ \text{plot} \left( f, \pi \frac{4}{4} .. \pi \right) \]

We define a continuous function

And compute its derivative symbolically.

We generate a plot to get an idea of where to look for minima and maxima. Evidently the minimum is in the interior of the interval and the maximum is at the left.
Finding where the derivative is zero should produce the minimum.

The minimum is confirmed to be zero -- the graph seems to indicate that but the picture doesn't let us know whether it's really zero or just really small.

Standard calculus procedure says to check out the value of the end points for the maximum. One is clearly larger than the other, and is bigger than the place within where the derivative is zero.

Thus we conclude that $f$ attains a maximum at $x = -\frac{\pi}{4}$. The value there is $2 + \sqrt{2}$. The minimum is zero, at $x = \frac{\pi}{2}$.

Example 12.1.1 Finding the minimum and maximum using textually-specified commands

Anton, Calculus, 8th ed. Problem 307
Find the absolute minimum of 
\[ (x^2 + x + 5)^{\frac{2}{3}} \text{ for } x \text{ in } [-2..3] \]
\[ (x^2 + x + 5)^{\frac{2}{3}} \rightarrow \]

We enter the expression, and plot it on -2..3 using the 2-D plot builder.

We do a chain of calculations to find what is obviously the minimum point.
Compare to example 12.1.2 where because we assigned results to variables, we could just type in the name of the variable again. Here we have to copy the expression.

We had to copy the expression again, and type in "-1/2" to calculate the minimum value.
Once we have a number, we can also see approximately what it is.
Section 12.2 One-step extrema finding using \texttt{maximize} and \texttt{minimize}

Maple also provides both exact and approximate numerical functions that do the entire sequence of steps needed to find extrema. In this section, we discuss the Maple functions that provide \textit{exact} solutions to such problems.

The Maple function for finding maxima is, appropriately enough, called \texttt{maximize}. If \texttt{expr} is an expression involving a variable \texttt{x}, then \texttt{maximize(expr,x=range)} will produce maximum value of the expression for \texttt{x} in that range. If there is no maximum value, or if Maple can't find it, NULL is returned. Thus if \texttt{maximize} returns an answer, it should be "believable", but the absence of an answer doesn't mean that there is no answer. \texttt{infinity} or \texttt{-infinity} may be an answer if the function is unbounded within the range.

It is often a good idea when working on extrema problems to do a plot of the expression in question, so that you can some notion of where the maxima will be and what their values will be like.

\textbf{Example 12.2.1 Finding the minimum and maximum using textually-specified commands}

From Anton, Calculus 8th ed. p. 307 problem 13 (modified)

Find the absolute maximum and minimum values of

\[
f := (x) \rightarrow 2 - 2 \cdot \sin(x) \quad \text{for} \quad x \in \frac{\pi}{4} \ldots \pi
\]

\textbf{Solution}

\[
f := (x) \rightarrow 2 - 2 \cdot \sin(x) \quad \text{for} \quad x \rightarrow 2 - 2 \sin(x)
\]

(1.2.1) We define a function

We generate a plot to get an idea of where to look for minima and maxima. Evidently the minimum is in the middle and the maximum is at the left.
smallestValue := minimize\( f(x), x = -\frac{\pi}{4} \ldots \pi \) \\
\quad 0 \quad (1.2.2)

largestValue := maximize\( f(x), x = -\frac{\pi}{4} \ldots \pi \) \\
\quad \quad 2 + \sqrt{2} \quad (1.2.3)

The minimum is confirmed to be zero -- the graph seems to indicate that but the picture doesn't let us know whether it's really zero or just really small. Note again that this is the minimum value, it is not the location of the minimum value.

Finding the maximum is a copy/edit job once we've worked out how to do the minimum.

Sometimes, you want not only what the maximum or minimum value is, but also where it is. If you give maximize or minimize an extra third parameter, location, then it will return a sequence for a result. This sequence has a logical but somewhat intricate structure. You can use a chain of selections to extract the location var= location point from the answer given by Maple.

**Example 12.2.2 Finding the minimum and maximum using textually-specified commands**

From Anton, Calculus 8th ed. p. 307 problem 13 (modified)

Find the absolute maximum and minimum values of
We define a function

\[ f := (x) \rightarrow 2 - 2 \cdot \sin(x) \text{ for } x \text{ in } -\frac{\pi}{4} .. \pi \]

**Solution**

\[ f := (x) \rightarrow 2 - 2 \cdot \sin(x) \]

\[ x \rightarrow 2 - 2 \sin(x) \tag{1.2.4} \]

\[ \text{plot}(f, -\frac{\pi}{4} .. \pi) \]

We generate a plot to get an idea of where to look for minima and maxima. Evidently the minimum is in the middle and the maximum is at the left.

\[ \text{smallestValue} := \text{minimize}(f(x), x = -\frac{\pi}{4} .. \pi) \]

\[ 0 \tag{1.2.5} \]

\[ \text{minResult} := \text{minimize}(f(x), x = -\frac{\pi}{4} .. \pi, \text{location}) \]

\[ 0, \left\{ \left[ \left\{ x = \frac{1}{2} \pi \right\}, 0 \right] \right\} \tag{1.2.6} \]

This is what we did in Example 12.2.1 -- we just get the smallest value of \( f \).

\[ \text{select entry 2} \]

\[ \left\{ \left[ \left\{ x = \frac{1}{2} \pi \right\}, 0 \right] \right\} \]

\[ \text{select entry 1} \]

\[ \left\{ x = \frac{1}{2} \pi \}, 0 \right\} \]

\[ \text{select entry 1} \]

\[ x = \frac{1}{2} \pi \]

\[ \text{right hand side} \]

\[ \frac{1}{2} \pi \]

This is what minimize does with the extra parameter \textit{location}.

\[ \text{select entry 1} \]

\[ \text{select entry 1} \]

\[ x = \frac{1}{2} \pi \]

\[ \text{right hand side} \]

\[ \frac{1}{2} \pi \]

This longwinded sequence of selections is needed to get just the location of the minimum.

\[ \text{minLocation} := \text{rhs}([\text{minResult}][2][1][1][1]) \]

\[ \frac{1}{2} \pi \tag{1.2.7} \]

This is a situation where the textual form of the command to produce the result is about the same level of difficulty to enter as the
maxResult := maximize\left(f(x), x = -\frac{\pi}{4} \ldots \pi, \text{location} \right)

\[ 2 + \sqrt{2}, \left\{ \left\{ x = -\frac{1}{4} \pi, 2 + \sqrt{2} \right\} \right\} \]  \hspace{1cm} (1.2.8)

maxLocation := rhs(maxResult[2][1][1][1])

\[ -\frac{1}{4} \pi \]  \hspace{1cm} (1.2.10)

Finding the maximum is a copy/edit job once we've worked out how to do the minimum.

If you want to evaluate another expression at a minimum point, then eval(expr, result[2][1][1]) will do that. This is because result[2] contains a set which describes the location/value pair. result[2][1] returns the first element of that, which is a list containing a particular location/value pair. result[2][1][1] is a set of the form \{var = extreme value\}. eval(expr, \{var=value\}) will evaluate expr with value replacing all occurrences of var in expr.

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**Example 12.2.3 Finding the minimum and maximum using textually-specified commands**

From Anton, Calculus 8th ed. p. 316-317

Example 7

A liquid form of penicillin manufactured by a pharmaceutical firm is sold in bulk at a price of $200 per unit. If the total production cost (in dollars) for \(x\) units is

\[ C(x) = 500000 + 80 \cdot x + 0.003 \cdot x^2 \]

and if the production capacity of the firm is at most 30,000 units in a specified time, how many units of penicillin must be manufactured and sold in that time to maximize the profit? What will the revenue be at the maximal profit?

**Solution**

\[ C := (x) \rightarrow 500000 + 80 \cdot x + 0.003 \cdot x^2 \]  \hspace{1cm} (1.2.11)

\[ R := (x) \rightarrow 200 \cdot x \]  \hspace{1cm} (1.2.12)

We define a Cost, Revenue, and Profit function, using the information given in the problem about cost, and the standard definitions from economics for revenue and profit.
We plot the profit function to scope out its behavior. Evidently the function does hit a maximum, roughly around $x=20000$. If we got $x=35$ as the answer from our *maximum* calculation in Maple, we would wonder if we did things correctly.

We are getting floating point answers from Maple because some of the numbers in $C$ are floating point. If we wanted an exact calculation, we would change the $0.003$ into $3/1000$.

We learned from Example 12.2.2 how to get the location information out. We use the *floor* function (see Chapter XX) to round down to the nearest integer, since we can't produce a fraction of a unit.

We used the clickable menu item "Numeric Formatting" to reformat the result in terms of currency.
Section 12.3 One-step approximations to extrema using Optimization[Maximize] and Optimization[Minimize]

Maple has `solve` and `fsolve` that provide exact and approximate solutions. Similarly there is Optimization[Maximize] and Optimization[Maximize] play the role of `fsolve` to maximize and minimize. The clickable interface's "Optimization" operation also invokes the approximate optimizers.

Optimization[Maximize] and Optimization[Minimize] always return a list. The first item of the list is the extreme value, and the second item in the list is a list of equations describing the location of the extreme value. As with the exact optimizing functions maximize and minimize, the second item can be used as a parameter to eval to evaluate an expression at the location of an extreme value.

Example 12.3.1 Finding an approximation to the minimum with Optimization[Minimize]

Anton, Calculus 8th ed. problem 47, p. 321 (modified)

Two particles, A and B, are in motion in the \(xy\)-plane. Their coordinates at each instant of time \(t \geq 0\) are given by \(x_A = t, y_A = 2 \cdot t, x_B = 1 - t, y_B = t\). Find the minimum distance between \(A\) and \(B\).

\[
\begin{align*}
x_A & := t & (1.3.1) \\
y_A & := 2 \cdot t & (1.3.2) \\
x_B & := 1 - t & (1.3.3) \\
y_B & := t & (1.3.4)
\end{align*}
\]

\(p1 := [x_A, y_A] \quad [t, 2 \cdot t] \quad (1.3.5)\)

\(p2 := [x_B, y_B] \quad [1 - t, t] \quad (1.3.6)\)

\[
\text{dist} := (LI, L2) \rightarrow \sqrt{\left((LI[1] - L2[1])^2 + (LI[2] - L2[2])^2\right)}
\]

\((LI, L2) \rightarrow \quad (1.3.7)\)

We create data structures that are a list of the \(x\) and \(y\) positions for each point.

We create the objective function, which is the Euclidean distance between the two points.

This is what the expression to be minimized looks like for the motion expressions we were given in the problem. It's a function of \(t\).

Our first guess about looking for the minimum doesn't really give us enough detail about the
\[ \sqrt{(L1_1 - L2_1)^2 + (L1_2 - L2_2)^2} \]
\[
\text{dist}(p1, p2) = \sqrt{5 t^2 - 4 t + 1} \tag{1.3.8}
\]
\[ \text{plot}(\text{dist}(p1, p2), t = 0 .. 10) \]

\[ \text{plot}(\text{dist}(p1, p2), t = 0 .. 2) \]

\[
\text{with(Optimization)} \tag{1.3.9}
\]
\[ \text{[ImportMPS, Interactive, LP	ext{Solve}, LSSolve, Maximize, Minimize, NLPSolve, QPSolve]} \]
\[ \text{minResult} := \text{Minimize}(\text{dist}(p1, p2), t = 0 .. 2) \tag{1.3.10}\]
\[ [0.44721359549958150, [t = 0.400000000000000022]] \]
\[ \text{minLocation} := \text{minResult}[2] \tag{1.3.11}\]
\[ [t = 0.400000000000000022] \]
\[ \text{ALocation} := \text{eval}(p1, \text{minLocation}) \tag{1.3.12}\]
\[ [0.400000000000000022, 0.8000000000] \]
\[ \text{BLocation} := \text{eval}(p2, \text{minLocation}) \]

It's easy enough to use the visual feedback of the first plot to come up with the second plot range that's more suitable. In other words, we determined the new range 0..2 "by eyeball".

We load the entire Optimization package in.... so we can invoke the minimizer as Minimize rather than as Optimization [Minimize]. Note that the exact behavior of the function near the minimum.
location for \( t \) is 0.4, but that rounding error made during the operation of Minimize leads to only an approximation of the exact answer. This is typical.

According to the structure returned by this function, the location information is in the second element of the result.

We can find the location of particles A and B at \( t=0.4 \) by evaluating the expression \( p1 \) and \( p2 \) at that value of \( t \). Note that \texttt{minLocation} is a list consisting of an equation, which \texttt{eval} can use as its second argument.

\begin{table}[h]
\centering
\begin{tabular}{|l|l|}
\hline
\textbf{Operation} & \textbf{Example} \\
\hline
Exact minimum or maximum value & \( f := (x) \rightarrow 2 - 2 \cdot \sin(x) \) for \( x \) in \( \frac{\pi}{4} .. \pi \) \\
\hline
\end{tabular}
\end{table}

\begin{equation}
\text{Solution}
\end{equation}

\( f := (x) \rightarrow 2 - 2 \cdot \sin(x) \)
\( x \rightarrow 2 - 2 \sin(x) \) (1.4.1)

\texttt{plot}(f, \frac{\pi}{4} .. \pi) \text{ It's usually a good idea to plot the objective function to get some sense of where the extrema are. Remember that the function to be investigated should be continuous if you expect good results from calculus techniques.}
smallestValue := minimize \( f(x), x = -\frac{\pi}{4} \ldots \pi \) 
\[0\] \hspace{1cm} (1.4.2)

largestValue := maximize \( f(x), x = -\frac{\pi}{4} \ldots \pi \) 
\[2 + \sqrt{2}\] \hspace{1cm} (1.4.3)

Exact minimum or maximum value location

\[minResult := minimize \left( f(x), x = -\frac{\pi}{4} \ldots \pi, \text{location} \right)\]
\[0, \left\{ \left\{ x = \frac{1}{2} \pi \right\}, 0 \right\} \] \hspace{1cm} (1.4.4)

\[maxResult := maximize \left( f(x), x = -\frac{\pi}{4} \ldots \pi, \text{location} \right)\]
\[2 + \sqrt{2}, \left\{ \left\{ x = -\frac{1}{4} \pi \right\}, 2 + \sqrt{2} \right\} \] \hspace{1cm} (1.4.5)

Evaluation of an function at an extremum point

\[g := (t) \rightarrow \sin(2 \cdot t)\] Evaluate \( g \) at the location of the smallest value of \( f \).
\[t \rightarrow \sin(2 \cdot t)\] \hspace{1cm} (1.4.6)

\[minLocation := \text{rhs}(\text{minResult}[2][1][1][1])\]
\[\frac{1}{2} \pi\] \hspace{1cm} (1.4.7)

\[g(minLocation)\]
\[0\] \hspace{1cm} (1.4.8)

Evaluation of an expression at an

\[expr := x^2 + \sin(x)\]
<table>
<thead>
<tr>
<th>Extremum point</th>
<th>( x^2 + \sin(x) ) \hspace{1cm} (1.4.9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{minEvalpt} := \text{minResult}[2][1][1] )</td>
<td>( \left{ x = \frac{1}{2} \pi \right} ) \hspace{1cm} (1.4.10)</td>
</tr>
<tr>
<td>( \text{eval(expr, minEvalpt)} )</td>
<td>( \frac{1}{4} \pi^2 + 1 ) \hspace{1cm} (1.4.11)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Approximate minimum or maximum value and location</th>
<th>( \text{approxMinResult} := \text{Optimization}[\text{Minimize}] \left( f(x), x = \right. )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( -\frac{\pi}{4} \ldots \pi )</td>
</tr>
<tr>
<td></td>
<td>( [0., [x = 1.57079632679489478]] ) \hspace{1cm} (1.4.12)</td>
</tr>
<tr>
<td>( \text{approxMaxResult} := \text{Optimization}[\text{Maximize}] \left( f(x), x = \right. )</td>
<td>( -\frac{\pi}{4} \ldots \pi )</td>
</tr>
<tr>
<td></td>
<td>( [3.41421354582206549, [x = )</td>
</tr>
<tr>
<td></td>
<td>( -0.785398151694103208]] ) \hspace{1cm} (1.4.13)</td>
</tr>
<tr>
<td>( g(\text{rhs(approxMaxResult}[2][1])) )</td>
<td>( -1.000000000 )</td>
</tr>
<tr>
<td>( \text{eval(expr, approxMaxResult[2])} )</td>
<td>( -0.0902565162 ) \hspace{1cm} (1.4.14)</td>
</tr>
</tbody>
</table>