Chapter 9 More Mathematical computation

Section 9.1  Solving multiple equations and/or inequalities

Maple can solve systems of equations by giving `solve` a set of equations and a set of variables. One can also enter inequalities instead of equations.

`fsolve` can handle systems of equations, but it does not handle inequalities.

Maple's "firmest" solution power comes from the exact solution of systems of linear equations. The solution, if one exists, will be described in full. If NULL is returned as the answer from `solve`, it means that no solution exists.

Exact solution of inequalities may not be successful even if there is a solution. That is, if Maple returns NULL in this situation, it does not necessarily mean that there is no solution, it may only mean that Maple couldn't find any solutions.

If Maple finds an approximate solution to a system of equations, it does not necessarily mean that a solution exists, just that the approximation technique (which are not infallible) thinks that it's found one. Even if a solution exists to a system, an approximate solution may or may not be close to the exact solution, due to accumulated rounding errors resulting from the use of floating point (limited precision) numbers. This limitation exists in most systems (e.g. Matlab, Fortran or C++) packages that use floating point arithmetic to solve systems of equations.

<table>
<thead>
<tr>
<th>Example 8.1.1 solve and fsolve with systems of equalities or inequalities</th>
<th>Commentary</th>
</tr>
</thead>
</table>
| `solve`( {3 \cdot x + y = 5, 4 - y = 7 \cdot x}, \{x, y\}`)
` \begin{align*}
  x &= \frac{1}{4}, \\
  y &= \frac{23}{4}
\end{align*}`
| To solve a system of equations, give `solve` a set of equations and a set of variables. The result can be a set of equations. `solve` returns NULL if it can't any solution. |
| `fsolve`( {3 \cdot x + y = 5, 4 - y = 7 \cdot x}, \{x, y\}`)
` (x = -0.2500000000, y = 5.750000000)`
| Maple may return expressions involving more advanced mathematics in some cases. If you get back an answer that you don't comprehend, it's a sign that either you've asked the wrong problem by mistake, or that you're going to have to get more help in understanding the situation. |
| `solve`( \{x + y = 3, x + y = 4\}, \{x, y\}`)
` \begin{align*}
  x &= \text{RootOf}(-f(f(_Z)) + _Z), \\
  y &= f(\text{RootOf}(-f(f(_Z)) + _Z))
\end{align*}`
| We are interested in finding the radius of a |
\begin{align*}
\left\{ r = \frac{\sqrt{10}}{\sqrt{\pi}} \right\} & \quad (1.1.4) \\
\text{eval}(2 \cdot \pi \cdot r, (1.1.4)) & \quad (1.1.5) \\
2 \sqrt{\pi \cdot 10} & \quad (1.1.6)
\end{align*}

circle whose area is 10. In order to discard negative roots of the quadratic equation, we put in the additional inequality that $r$ is positive. We get a set back as the answer.

We can use the result from solve as is, to figure out the circumference of the specified circle, both exactly, and as a five digit approximation.

\begin{align*}
solve((x + y > 5, x - y < 3), \{x, y\}) & \quad (1.1.7) \\
(x \leq 4, -x + 5 < y), (4 < x, x - 3 < y)
\end{align*}

fsolve((x + y > 5, x - y < 3), \{x, y\})

Error, Got internal error in Typesetting:-Parse : 
"_Inert_DELAYLESSTHAN' is not a valid inert form"

\begin{align*}
solve([3 \cdot x + y = 5, 4 - y = 2 \cdot x], \{x, y\}) & \quad (1.1.8) \\
[[x = 1, y = 2]]
\end{align*}

fsolve([3 \cdot x + y = 5, 4 - y = 2 \cdot x], \{x, y\})

Error, (in fsolve) invalid arguments

\begin{align*}
solve(\{x^2 + 2 \cdot x - 5 = 0, x > 0\}, \{x\}) & \quad (1.1.9) \\
\{x = -1 + \sqrt{6}\}
\end{align*}

fsolve(\{x^2 + 2 \cdot x - 5 = 0, x > 0\}, \{x\})

Error, (in fsolve) expecting an equation or set or list of equations, but received inequality \{x^2+2x-5 = 0, 0 < x\}

solve can handle systems of inequalities. Here, the solution is described as a sequence of two different sets.

fsolve can't handle inequalities. Inequalities aren't that easily handled through approximate calculations, because it's not uncommon for rounding error to seriously affect the accuracy of the results.

Lists of equations and variables sometimes work, too. But then the result comes back in a different form: a list of solutions. Each solution is expressed as a list of equations for the variables involved in the system.

fsolve doesn't accept lists of unknowns, at least in Maple 12. There's no strong reason why the design has to be this way, but that's Maple 12's fsolve rejects parameters of type list.

Sometimes you can give extra constraints on the solution through the use of inequalities

fsolve doesn't work with inequalities
If the result of a multi-variate `solve` is a piecewise expression, then the additional phrase `assuming inequality` can eliminate some of the pieces.

**Example 9.1.2 Use of assuming**

If the result of a multi-variate `solve` is a piecewise expression, then the additional phrase `assuming inequality` can eliminate some of the pieces.

<table>
<thead>
<tr>
<th>Solve (x^2 = 5 \cdot \alpha, x &gt; 0), {x}</th>
<th>We get a piecewise expression as the result of solve, if you don't put any restrictions on alpha.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solve (x^2 = 5 \cdot \alpha, x &gt; 0), {x} assuming \alpha &gt; 0</td>
<td>Adding information that alpha is positive removes some of the pieces, and greatly simplifies the information being presented.</td>
</tr>
</tbody>
</table>

**Section 9.2 Calculus: differentiation, simplification**

As it is taught in traditional first-year calculus, differentiation is an operation on functions. Maple knows how to differentiate all of the common functions found in calculus. It is particularly useful for performing differentiation when it would take a lot of algebraic manipulation to do the operations by hand.

`diff` as a function takes two or more arguments. The first argument must be (or evaluate to) an expression. The second argument must be (or be an expression that evaluates to) the variable of differentiation. The there are third, fourth, etc. arguments provided, they are used as variables of higher derivatives.

The result of `diff` is an expression or a number if the derivative is a numerical constant. Evaluation of a derivative expression can occur through `eval` as with other expressions.

**Example 8.2.1 Symbolic differentiation and evaluation of derivatives**

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>expr := x^2 - 2 \cdot x + 5</code></td>
<td>Find the first derivative of the expression with respect to x.</td>
</tr>
<tr>
<td><code>diff(expr, x)</code></td>
<td>(1.2.1)</td>
</tr>
<tr>
<td><code>2 \cdot x - 2</code></td>
<td>(1.2.2)</td>
</tr>
<tr>
<td><code>posExpr := \sin(\omega \cdot t + 5)^2</code></td>
<td>Find the first derivative of the expression with</td>
</tr>
<tr>
<td><code>\sin(\omega \cdot t + 5)^2</code></td>
<td>(1.2.3)</td>
</tr>
<tr>
<td><code>diff(posExpr, t)</code></td>
<td>(1.2.4)</td>
</tr>
</tbody>
</table>
\[
2 \sin(\omega t + 5) \cos(\omega t + 5) \omega 
\]
\[
diff(posExpr, t, t)
\]
\[
2 \cos(\omega t + 5)^2 \omega^2 - 2 \sin(\omega t + 5)^2 \omega^2
\]
\[
diff(expr, t)
\]
\[
0
\]

**Example 8.2.2 Plotting a function and its derivative together**

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f := (a, b) \to \sin(a) + \cos(b) )</td>
<td>We're already used to functions that take one or two arguments. Note that unlike mathematical convention, in programming variable names often do have names that are more than one letter long. This is to increase intelligibility to readers looking at the programming. Although it seems minor, ease of comprehension can play an significant role.</td>
</tr>
<tr>
<td>( g := (\alpha) \to f \left( \frac{\alpha}{2}, \frac{\alpha}{3} \right) )</td>
<td>We're already used to functions that take one or two arguments. Note that unlike mathematical convention, in programming variable names often do have names that are more than one letter long. This is to increase intelligibility to readers looking at the programming. Although it seems minor, ease of comprehension can play an significant role.</td>
</tr>
</tbody>
</table>
In the cost of developing and using software, so is an important engineering concern.

In this example, we plot a "Strange" function built out of trigonometric parts, and plot it and its derivative. Since g is a function g(t) will evaluate to the expression. Thus the plotting variable should be $t$ rather than $\alpha$ or some other variable.

\[
\begin{align*}
\text{fderiv} & := \text{diff}(g(\alpha), \alpha) \\
\alpha & = \frac{1}{2} \cos\left(\frac{1}{2} \alpha\right) - \frac{1}{3} \sin\left(\frac{1}{3} \alpha\right) \\
\text{plot}(\{g(\alpha), \text{fderiv}\}, \alpha = 0 .. 30)
\end{align*}
\]

The value of $\text{fderiv}$ is an expression involving $\alpha$.

We plot two expressions involving the variable $\alpha$ on the same plot. The use of a set ({ }) as a first argument to $\text{plot}$ to do multiple plots was first explained in section 6.3.

We need to use $\alpha$ as the plotting variable here since the value of $\text{fderiv}$ is an expression involving $\alpha$.

\section*{Section 9.3 Calculus: limits}

You can use the clickable interface to compute limits by selecting the appropriate item from the Expression palette and then filling in the template as needed. Maple uses calculus techniques (e.g. l'Hôpital's Rule) to compute limits symbolically.
Example 9.3.1 Clickable interface version of limits

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lim_{x \to 3} \frac{1}{(x - 3)^2} ) ( \infty ) ( (1.3.1) )</td>
<td>The result of ( \text{limit} ) can be an expression (possibly involving positive or negative infinity) as well as the symbol \textit{undefined}.</td>
</tr>
<tr>
<td>( \lim_{x \to -\infty} \frac{\sin(x)}{x} ) ( 0 ) ( (1.3.2) )</td>
<td>Even though the limit exists as ( t ) approaches from the right or left, they do not agree, so there is no &quot;two-sided&quot; limit.</td>
</tr>
<tr>
<td>( \lim_{t \to 0} \frac{1}{t} ) ( \text{undefined} ) ( (1.3.3) )</td>
<td>To take a one-sided limit, add a &quot;+&quot; or &quot;-&quot; superscript to the limit point.</td>
</tr>
<tr>
<td>( \lim_{t \to 0^-} \frac{1}{t} ) ( -\infty ) ( (1.3.4) )</td>
<td>Note that the value returned as the limit for ( x = a ), while true for most values of ( a ), is not really valid for ( a=0 ).</td>
</tr>
<tr>
<td>( \lim_{x \to a} \frac{\sin(x)}{x} ) ( \frac{\sin(a)}{a} ) ( (1.3.5) )</td>
<td>The textual version of taking limits involves the \textit{limit} function. \textit{limit} takes at least two arguments. The first argument is the expression that you wish to take the limit of. The second argument is an equation indicating the limit variable and the limiting value. If you supply a third argument, it indicates whether a &quot;right sided&quot;, &quot;left sided&quot; limit is desired instead of a two-sided limit.</td>
</tr>
</tbody>
</table>

The textual version of taking limits involves the \textit{limit} function. \textit{limit} takes at least two arguments. The first argument is the expression that you wish to take the limit of. The second argument is an equation indicating the limit variable and the limiting value. If you supply a third argument, it indicates whether a "right sided", "left sided" limit is desired instead of a two-sided limit.

Example 9.3.1 Textual version of limits

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{limit}(1/(x-3)^2, x = 3) ) ( \infty ) ( (1.3.6) )</td>
<td>The first argument is the expression that you wish to take the limit of. The second argument is an equation indicating the limit variable and the limiting value.</td>
</tr>
<tr>
<td>( \text{limit}((\sin(x))/x, x = \infty) ) ( 0 ) ( (1.3.7) )</td>
<td>You can use &quot;infinity&quot; or &quot;-infinity&quot; as the way of specifying ( \infty ) or (-\infty ) textually without use of the palette.</td>
</tr>
<tr>
<td>( \text{limit}(1/t, t = 0, \text{left}) ) ( -\infty ) ( (1.3.8) )</td>
<td>The third optional third argument to \textit{limit} can specify a one sided limit. This is the textual</td>
</tr>
</tbody>
</table>
One could make a case that the second argument for the textual form of \textit{limit} should be something looking like \textit{var} -> \textit{limit value} since that is the more conventional terminology. But because of the limitations of Maple's processing capabilities for its programming language, it would be more expensive to support arrows for both this meaning in limits and the use of \textit{-} in function definitions. So users must get used to using equations rather than the standard math symbol for "approaches".

\section*{Section 9.4 For the advanced: multivariate differentiation.}

\textit{This section can be skipped until you take multivariate calculus.}

Calculations involving expressions or functions of several variables are a situation where having a computer to do the symbolic calculation is even more useful. One can apply differentiation several times via the clickable interface, but this can become tedious when computing higher derivatives. The textual interface can be faster. Other multivariate operations have no clickable

\begin{center}
\begin{tabular}{|l|l|}
\hline
\textbf{Examples 9.4.1 Multivariate calculus operations} & \\
\hline
Let \( z = \sqrt{x} \cdot \cos(y) \). Compute \( z_{xx} \), \( z_{yy} \), \( z_{xy} \) and \( z_{yx} \). & This problem 67 from Anton, Calculus 8th edition, section 14.3. \\
\hline
\textit{zexpr} := \sqrt{x} \cdot \cos(y) & \text{We enter the expression and give it a name.} \\
\hline
\textit{diff (zexpr, x, x)} & \text{We use the textual version of differentiation to compute} \ \frac{\partial^2}{\partial x^2} \text{ of the expression.} \\
& \text{This computes} \ \frac{\partial}{\partial y} \frac{\partial }{\partial x} = \frac{\partial^2}{\partial x \partial y} \text{ of the expression.} \\
& \text{This computes} \ \frac{\partial}{\partial x} \frac{\partial }{\partial y} = \frac{\partial^2}{\partial y \partial x} \text{ of the expression.} \\
\hline
\end{tabular}
\end{center}
\begin{align*}
\text{diff (zexpr, y, x)}
&= \frac{-1}{2} \frac{\sin(y)}{\sqrt{x}} \\
\sqrt{x} \cdot \cos(y) \quad \text{differentiate w.r.t. } x
&= \frac{1}{2} \frac{\cos(y)}{\sqrt{x}} \\
\text{differentiate w.r.t. } y
&= \frac{-1}{2} \frac{\sin(y)}{\sqrt{x}}
\end{align*}

expression. Since the expression describes a continuous function of \(x\) and \(y\), we get the same results as when we took the partial derivatives in the reverse order.

This is one situation where the textual interface gives the answer faster than the clickable interface. However, the clickable interface does display the intermediate results.

\begin{align*}
\text{Given that } x^3 + y^2 \cdot x - 3 &= 0, \text{ find } \frac{dy}{dx}. \\
\text{implicitdiff } (x^3 + y^2 \cdot x - 3 = 0, y, x)
&= \frac{-1}{2} \frac{3 x^2 + y^2}{y \cdot x} \\
\text{Given } y \cdot e^x \cdot \sin(3 \cdot z) = 3 \cdot z, \text{ compute } \frac{\partial}{\partial x} z \text{ and } \frac{\partial}{\partial y} z \text{ through implicit differentiation.}
\end{align*}

This is Example 3 from section 14.5 of Anton, Calculus 8th edition.

Not surprisingly, Maple has a built-in command to do implicit differentiation. You can read more about it by typing "implicit differentiation" or "implicitdiff" into Maple's on-line help.

\begin{align*}
\text{This is problem 43 in section 14.5 of Anton.}
\end{align*}

The clickable interface lets us select what the dependent variable is, and which is the variable of differentiation. Here, we pick "z" and 'x".

Although the output does not show it, we picked "z" and "y" here. Using the textual version implicitdiff would clearly show that different things are being computed.

\begin{align*}
\text{Find the directional derivative of } f(x, y) = \sqrt{x \cdot y \cdot e^x} \text{ at } P(1, 1) \text{ in the direction of the negative } y\text{-axis.}
\end{align*}

This is problem 25 from section 14.6 of Alton, Calculus 8th edition.

This loads all the functions from the Student [MultivariableCalculus] package. You can read more about this by entering "directional derivative" into Maple's on-line help.
With the package loaded in, we can use the DirectionalDerivative function. The second argument is the point where we are evaluating the derivative, and the third argument describes the direction. We need to express this direction as a "unit vector".

This 3-d graph describing the situation with the directional derivative of 1.4.8 was produced by selecting Tools->Tutors->Multivariable Calculus -> Directional Derivative in the Maple interface and supplying the information (in textual form) described in that problem.

### Section 9.5 For the advanced: integration

This section can be skipped until you need to do integrals.

Maple can perform symbolic integrals about as well as most people can. Because it is indefatigably rigorous about doing algebra, it can often get an answer more reliably, at least for longer problems. We will discuss symbolic integration further in a few chapters, but will preview it here. You can read more about this by entering "integration" into Maple's on-line help. Next term we will work more with symbolic integration, but since we are introducing calculus in this chapter, we wanted to give a more complete picture of Maple's calculus capabilities.

**Example 9.5.1 Symbolic Integration**

\[
\int \sin\left( \frac{x}{2} \right) \cdot x^2 \, dx
\]

\[
\int_0^{\frac{\pi}{4}} \ln(x) \cdot x \, dx
\]

\[
- \frac{1}{16} \pi^2 \ln(2) + \frac{1}{32} \pi^2 \ln(\pi) \quad (1.5.1)
\]

Maple can handle do indefinite and definite integration symbolically using the integration item from the Expression palette.
The clickable interface has an "integrate" item. You have to select the variable of integration. The clickable interface does not have a "definite integration" menu item since it is too much work to enter the information in that fashion. You can

\[
\sin\left(\frac{x}{2}\right) x^2 \int \text{w.r.t. } x
\]

\[
-2 \cos\left(\frac{1}{2} x\right) x^2 + 16 \cos\left(\frac{1}{2} x\right)
+ 8 \sin\left(\frac{1}{2} x\right) x
\]

The function that does integration is called \textit{int} (although it could have been called \textit{integrate}, that doesn't work in Maple). The first argument is the expression to be integrated, the second is either the variable of integration (for indefinite integration), or an equation stating the variable and a range (for definite integration).

\[
\text{int}\left(\sin\left(\frac{x}{2}\right) x^2, x\right)
- 2 \cos\left(\frac{1}{2} x\right) x^2 + 16 \cos\left(\frac{1}{2} x\right)
+ 8 \sin\left(\frac{1}{2} x\right) x
\]

\[
\text{int}\left(\ln(x) \cdot x, x = 0 .. \frac{\pi}{4}\right)
- \frac{1}{16} \pi^2 \ln(2) + \frac{1}{32} \pi^2 \ln(\pi)
- \frac{1}{64} \pi^2
\]

Maple has a number of tools for studying the part of calculus having to do with integration. For example, selecting Tools->Tutors->Calculus Single Variable ->Integration Methods will display a pop-up window where one can enter the textual form of an integrand and produce a step-by-step derivation of the symbolic integral, as illustrated by Figure 9.5.1 below.

\textbf{Figure 9.5.1 Symbolic Integration Tutor (from Tools->Tutors->Calculus Single Variable ->Integration Methods )}
Example 9.5.2  Numerical integration

\[
\int_0^{\pi/4} \tan(x^2) \, dx
\]

When Maple returns the same thing as what you entered, it means that it couldn't determine a simple expression for the integral.

When Maple returns the same thing as what you entered, it means that it couldn't determine a simple expression for the integral.

\[
\int_0^{\pi/4} \tan(x^2) \, dx = \frac{1}{4}\pi \int_0^{\pi/4} \tan(x^2) \, dx
\] (1.5.4)

However, it can determine that the area underneath the curve of \( \tan(x^2) \) between 0 and \( \pi/4 \) at 10 digits:

\[
\int_0^{\pi/4} \tan(x^2) \, dx \approx 0.1712289721
\]
\[ \text{evalf} \left( \int \tan(x^2), x = 0 .. \frac{\pi}{4}, 20 \right) \]

\[ 0.17122897213322970855 \] (1.5.5)

This uses the textual version of the integration and numerical approximation commands to approximate the area to 20 digits.

\[ \frac{\pi}{4} \] is approximately 0.1712. To do this, we used the "approximate to 10 digits" item of the clickable interface.

## Section 9.X Chapter summary

### Example 9.1.1 Solving equations

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{solve}( { 3 \cdot x + y = 5, 4 - y = 7 \cdot x }, { x, y }) ) [ { x = -\frac{1}{4}, y = \frac{23}{4} } ]</td>
<td>To solve a system of equations, give ( \text{solve} ) a set of equations and a set of variables. The result can be a set of equations. ( \text{solve} ) returns NULL if it can't any solution.</td>
</tr>
<tr>
<td>( \text{fsolve}( { 3 \cdot x + y = 5, 4 - y = 7 \cdot x }, { x, y }) ) ( (x = -0.2500000000, y = 5.750000000) ) (1.6.2)</td>
<td>Maple may return expressions involving more advanced mathematics in some cases. If you get back an answer that you don't comprehend, it's a sign that either you've asked the wrong problem by mistake, or that you're going to have to get more help in understanding the situation.</td>
</tr>
<tr>
<td>( \text{solve}( { x + y = 3, x + y = 4 }, { x, y }) )</td>
<td></td>
</tr>
<tr>
<td>( \text{solve}( { f(x) = y, f(y) = x }, { x, y }) ) ( (x = \text{RootOf}(-f(f(_Z)) + _Z), y = f(\text{RootOf}(-f(f(_Z)) + _Z)) ) (1.6.3)</td>
<td></td>
</tr>
</tbody>
</table>

### Example 9.1.2 Use of assuming

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
</table>
| \( \text{solve}( \{ x^2 = 5 \cdot \alpha, x > 0 \}, \{ x \}) \) \[
\begin{align*}
\text{if } & 0 < \sqrt{5} \sqrt{\alpha} \\
\{ x = \sqrt{5} \sqrt{\alpha} \} & \\
\text{if } & \sqrt{5} \sqrt{\alpha} < 0 \\
\{ x = -\sqrt{5} \sqrt{\alpha} \} & \\
\text{otherwise} & \\
\text{end if}
\end{align*}
\] (1.6.4) | We get a piecewise expression as the result of \( \text{solve} \), if you don't put any restrictions on alpha. Adding information that alpha is positive removes some of the pieces, and greatly |
| \( \text{solve}( \{ x^2 = 5 \cdot \alpha, x > 0 \}, \{ x \}) \) assuming \( \alpha > 0 \) | |

\( \text{solve} \) returns NULL if it can't any solution.
Example 8.2.1 Symbolic differentiation and evaluation of derivatives

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>${x = \sqrt{5} \sqrt{\alpha}}$</td>
<td>(1.6.5) simplifies the information being presented.</td>
</tr>
<tr>
<td>$expr := x^2 - 2 \cdot x + 5$</td>
<td>Find the first derivative of the expression with respect to $x$.</td>
</tr>
<tr>
<td>$diff (expr, x)$</td>
<td></td>
</tr>
<tr>
<td>$2x - 2$</td>
<td>(1.6.7)</td>
</tr>
<tr>
<td>$posExpr := \sin(\omega \cdot t + 5)^2$</td>
<td>Find the first derivative of the expression with respect to $t$.</td>
</tr>
<tr>
<td>$\sin(\omega t + 5)^2$</td>
<td>(1.6.8)</td>
</tr>
<tr>
<td>$diff (posExpr, t)$</td>
<td>Find the second derivative of the expression with respect to $t$. The result is the same as if we had taken the derivative of 1.2.4.</td>
</tr>
<tr>
<td>$2 \sin(\omega t + 5) \cos(\omega t + 5) \omega$</td>
<td>(1.6.9)</td>
</tr>
<tr>
<td>$diff (posExpr, t, t)$</td>
<td></td>
</tr>
<tr>
<td>$2 \cos(\omega t + 5)^2 \omega^2 - 2 \sin(\omega t + 5)^2 \omega^2$</td>
<td>(1.6.10)</td>
</tr>
<tr>
<td>$diff (expr, t)$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>(1.6.11)</td>
</tr>
<tr>
<td>$simplify((1.6.10))$</td>
<td></td>
</tr>
<tr>
<td>$2 \omega^2 \left(2 \cos(\omega t + 5)^2 - 1\right)$</td>
<td>(1.6.12)</td>
</tr>
<tr>
<td>$eval((1.6.7), x = 3)$</td>
<td>This is a way to compute $\frac{d}{dx} x^2 - 2 \cdot x + 5 \bigg</td>
</tr>
<tr>
<td>$4$</td>
<td>(1.6.13)</td>
</tr>
<tr>
<td>$eval((1.6.12), t = 47.0)$</td>
<td>This is a way to compute</td>
</tr>
<tr>
<td>$2 \omega^2 \left(2 \cos(47.0 \omega + 5)^2 - 1\right)$</td>
<td>(1.6.14)</td>
</tr>
</tbody>
</table>
\( \frac{d^2}{dt^2} \sin(\omega t + 5) \bigg|_{t = 47} \)  

The result of evaluation does not have to be a number even if a numeric value is being supplied for one of the variables in the expression being evaluated.

### Example 9.3.1 Clickable interface version of limits

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lim_{x \to 3} \frac{1}{(x - 3)^2} ) (\infty)</td>
<td>The result of limit can be an expression (possibly involving positive or negative infinity) as well as the symbol undefined.</td>
</tr>
<tr>
<td>( \lim_{x \to -\infty} \frac{\sin(x)}{x} ) (0)</td>
<td>Even though the limit exists as ( t ) approaches from the right or left, they do not agree, so there is no &quot;two-sided&quot; limit.</td>
</tr>
<tr>
<td>( \lim_{t \to 0} \frac{1}{t} ) undefined</td>
<td>To take a one-sided limit, add a &quot;+&quot; or &quot;-&quot; superscript to the limit point.</td>
</tr>
<tr>
<td>( \lim_{t \to 0^-} \frac{1}{t} ) (-\infty)</td>
<td>Note that the value returned as the limit for ( x = a ), while true for most values of ( a ), is not really valid for ( a = 0 ).</td>
</tr>
<tr>
<td>( \lim_{x \to a} \frac{\sin(x)}{x} = \frac{\sin(a)}{a} )</td>
<td></td>
</tr>
</tbody>
</table>

### Example 9.3.1 Textual version of limits

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>\texttt{limit}(1 / (x-3)^2, x = 3) (\infty)</td>
<td>The first argument is the expression that you wish to take the limit of. The second argument is an equation indicating the limit variable and the limiting value.</td>
</tr>
<tr>
<td>\texttt{limit}(\sin(x) / x, x = \infty)</td>
<td>You can use &quot;infinity&quot; or &quot;-infinity&quot; as the way of specifying ( \infty ) or (-\infty) textually without use of</td>
</tr>
</tbody>
</table>
$$\begin{align*}
0 & \quad (1.6.21) \\
\text{limit}(1 / t, t = 0, \text{left}) & = -\infty \quad (1.6.22) \\
\text{limit}(1 / t, t = 0, \text{right}) & = \infty \quad (1.6.23)
\end{align*}$$

The third optional third argument to \textit{limit} can specify a one sided limit. This is the textual way of specifying \( \lim_{t \to 0^-} \frac{1}{t} \). There is no simple way of specifying this.

This is the textual way of specifying \( \lim_{t \to 0^+} \frac{1}{t} \).