17 Calculus and optimization

17.1 Chapter Overview

One way that Maple and similar systems differ from calculators is that they can draw upon the hardware and storage resources to provide knowledge beyond arithmetic, being able to handle most of collegiate-level mathematics. We introduce the built-in functions for doing differentiation, limits, and finding minima and maxima. Like solve, the minima and maxima-finders come in two styles -- one that tries to calculate formulae for answers (exact calculation) and one that uses approximation techniques that work only when the answer is numeric.

17.2 Working with more mathematics

We have become accustomed to Maple's ability to do numerical calculation. Contrasting this with the kind of calculation available from today's calculators, we observe that it is different qualitatively from the calculation provided by a calculator because a) Programming (e.g. looping) allows us to do much larger quantities of calculation than we could through operations only through keystroke. b) Programming allows us to more easily reuse the calculation on varieties of similar problems. c) The document interface allows us to record both what we did and the results we got for future reference and/or use in presentations and documents d) We have taken advantage of some relatively powerful operations, such as solve which makes solving certain common problems much more convenient (even without the significant work to enter the expression). e) We have taken advantage of the more powerful visualization features available on a computer to enhance understanding of the phenomena being simulated. In addition to programmability and a more extensive user interface, computers in 2010 have another edge over calculators -- there is more room to store much more systems programming. Systems such as Maple, Mathematica and Matlab use this room to provide literally thousands of extra mathematical features. Some of it is immediately useful to first year university/college students in the form of symbolic computation -- results from calculus. Some of it involves operations and functions that you will hear about after the first year -- linear algebra, multivariate calculus, ordinary and partial differential equations, Laplace transforms, etc. Learning how to operate a system such as Maple gives you the understanding to pick up many of these extra features on your own, after you learn the additional mathematics. We give you a taste of this by exploring some of operations that are comprehensible with first year mathematics.

17.3 Differentiation, simplification

As it is taught in traditional first-year calculus, differentiation is an operation on functions. Maple knows how to differentiate all of the common functions found in calculus. It is particularly useful for performing differentiation when it would take a lot of algebraic manipulation to do the operations by hand.

diff as a function takes two or more arguments. The first argument must be (or evaluate to) an expression. The second argument must be (or be an expression that evaluates to) the variable of differentiation. The there are third, fourth, etc. arguments provided, they are used as variables of higher derivatives.

The result of diff is an expression or a number if the derivative is a numerical constant.

Evaluation of a derivative expression can occur through eval as with other expressions.

Symbolic differentiation using the expression palette

<table>
<thead>
<tr>
<th>Expression</th>
<th>diff w.r.t. x</th>
<th>Eval</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 - 2x + 5$</td>
<td>$2x - 2$</td>
<td>Enter the expression, and then right-click (control-click on Macintosh) to bring up the calculation options. Select differentiate-&gt;x</td>
<td></td>
</tr>
<tr>
<td>$\sin(\omega t + 5)$</td>
<td>$2\omega \cos(\omega t + 5)$</td>
<td>We can calculate the second derivative with respect to $t$ of this expression by performing differentiation twice. We simplify the expression a bit by performing the operation simplify-&gt;symbolic.</td>
<td></td>
</tr>
</tbody>
</table>
\[
\frac{d}{d\omega} \sin(\omega t + 5)^2 = 2 \sin(\omega t + 5) \cos(\omega t + 5) t
\]

If we want the document to display the mathematical notation for the derivative, we can select \( \frac{d}{dx} f \) from the expression palette and then fill in the slots \( f \) and \( x \). We can then get Maple to calculate the derivative by typing control– equals.

### Symbolic differentiation and evaluation of derivatives, textually

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>expr := x^2 - 2x + 5</code></td>
<td>Find the first derivative of the expression with respect to ( x ).</td>
</tr>
<tr>
<td><code>diff(expr, x)</code></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(17.1)</td>
</tr>
<tr>
<td><code>diff(posExpr := sin(\omega t + 5)^2)</code></td>
<td>Find the first derivative of the expression with respect to ( t ).</td>
</tr>
<tr>
<td><code>diff(posExpr, t)</code></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(17.3)</td>
</tr>
<tr>
<td><code>diff(posExpr, t, t)</code></td>
<td>Find the second derivative of the expression with respect to ( t ).</td>
</tr>
<tr>
<td></td>
<td>(17.5)</td>
</tr>
<tr>
<td><code>diff(expr, t)</code></td>
<td>diff's second argument must be the variable of differentiation. Note that if the variable doesn't occur in the expression, the derivative (according to the mathematical definition) is zero. If you are surprised by getting a zero derivative, check that you are using the correct variable to differentiate with respect to.</td>
</tr>
<tr>
<td></td>
<td>(17.6)</td>
</tr>
<tr>
<td><code>simplify(2 \cos(\omega t + 5)^2 \omega^2 - 2 \sin(\omega t + 5)^2 \omega^2)</code></td>
<td>simplify can reduce the size of the expression, although factor or expand sometimes work better. Sometimes additional trigonometric identities need to be applied, through simplify(...., trig).</td>
</tr>
<tr>
<td></td>
<td>(17.7)</td>
</tr>
<tr>
<td><code>eval(2x - 2, x = 3)</code></td>
<td>This is a way to compute ( \frac{d}{dx} x^2 - 2x + 5 ) ( x = 3 ).</td>
</tr>
<tr>
<td></td>
<td>(17.8)</td>
</tr>
<tr>
<td><code>eval(2 \omega^2 (2 \cos(\omega t + 5)^2 - 1), t = 47, 0)</code></td>
<td>This is a way to compute ( \frac{d^2}{dt^2} \sin(\omega t + 5)^2 ) ( t = 47 ). The result of evaluation does not have to be a number even if a numeric value is being supplied for one of the variables in the expression being evaluated.</td>
</tr>
<tr>
<td></td>
<td>(17.9)</td>
</tr>
</tbody>
</table>

### Plotting a function and its derivative together

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>f := (a, b) \rightarrow \sin(a) + \cos(b)</code></td>
<td>We're already used to functions that take one or two arguments. Note that unlike mathematical convention, in programming vari-</td>
</tr>
</tbody>
</table>
\[(a, b) \rightarrow \sin(a) + \cos(b) \quad (17.10)\]

Useful names often do have names that are more than one letter long. This is to increase intelligibility to readers looking at the programming. Although it seems minor, ease of comprehension can play an important role in the cost of developing and using software, so it is an important engineering concern.

\[g := \left( \alpha \rightarrow f\left( \frac{\alpha}{2}, \frac{\alpha}{3} \right) \right) \quad (17.11)\]

In this example, we plot a "strange" function built out of trigonometric parts, and plot it and its derivative. Since \( g \) is a function \( g(t) \) will evaluate to the expression \( \alpha \). Thus the plotting variable should be \( t \) rather than \( \alpha \) or some other variable.

\[\text{plot}(g(t), t = 0 \ldots 30)\]

\[f_{deriv} := \text{diff}(g(\alpha), \alpha)\]

\[\frac{1}{2} \cos\left( \frac{1}{2} \alpha \right) - \frac{1}{3} \sin\left( \frac{1}{3} \alpha \right) \quad (17.12)\]

The value of \( f_{deriv} \) is an expression involving alpha.

\[\text{plot}(\{g(\alpha), f_{deriv}\}, \alpha = 0 \ldots 30)\]

We plot two expressions involving the variable alpha on the same plot. The use of a set \( \{ \} \) as a first argument to \( \text{plot} \) to do multiple plots was first explained in section 6.3.
17.4 Limits

You can use the clickable interface to compute limits by selecting the appropriate item from the Expression palette and then filling in the template as needed. Maple uses calculus techniques (e.g. l'Hôpital's Rule) to compute limits symbolically.

### Clickable interface version of limits

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lim_{x \to 3} \frac{1}{(x-3)^2})</td>
<td>(\infty) (\quad) (17.13) The result of limit can be an expression (possibly involving positive or negative infinity) as well as the symbol undefined.</td>
</tr>
<tr>
<td>(\lim_{x \to a} \frac{\sin(x)}{x})</td>
<td>(0) (\quad) (17.14) Even though the limit exists as (t) approaches from the right or left, they do not agree, so there is no &quot;two-sided&quot; limit.</td>
</tr>
<tr>
<td>(\lim_{t \to 0} \frac{1}{t})</td>
<td>undefined (\quad) (17.15)</td>
</tr>
<tr>
<td>(\lim_{t \to 0^+} \frac{1}{t})</td>
<td>(-\infty) (\quad) (17.16) To take a one-sided limit, add a &quot;+&quot; or &quot;-&quot; superscript to the limit point.</td>
</tr>
<tr>
<td>(\lim_{x \to a} \frac{\sin(x)}{x})</td>
<td>(\frac{\sin(a)}{a}) (\quad) (17.17) Note that the value returned as the limit for (x=a), while true for most values of (a), is not really valid for (a=0).</td>
</tr>
</tbody>
</table>

The textual version of taking limits involves the limit function. limit takes at least two arguments. The first argument is the expression that you wish to take the limit of. The second argument is an equation indicating the limit variable and the limiting value. If you supply a third argument, it indicates whether a "right sided", "left sided" limit is desired instead of a two-sided limit.

### Textual version of limits

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>limit(1/(x-3)^2, x=3)</code></td>
<td>(\infty) (\quad) (17.18) The first argument is the expression that you wish to take the limit of. The second argument is an equation indicating the limit variable and the limiting value.</td>
</tr>
<tr>
<td><code>limit(sin(x)/x, x=\text{infinity})</code></td>
<td>(0) (\quad) (17.19) You can use &quot;infinity&quot; or &quot;-infinity&quot; as the way of specifying (\infty) or (-\infty) textually without use of the palette.</td>
</tr>
</tbody>
</table>
\[
\text{limit}(1/t,t=0,\text{left}) \quad -\infty \quad \text{(17.20)}
\]

The third optional third argument to \text{limit} can specify a one sided limit. This is the textual way of specifying \( \lim_{t \to 0^-} \frac{1}{t} \). There is no simple way of specifying this.

\[
\text{limit}(1/t,t=0,\text{right}) \quad \infty \quad \text{(17.21)}
\]

This is the textual way of specifying \( \lim_{t \to 0^+} \frac{1}{t} \).

One could make a case that the second argument for the textual form of \text{limit} should be something looking like \text{var} \rightarrow \text{limit value} since that is the more conventional terminology. But because of the limitations of Maple's processing capabilities for its programming language, it would be more expensive to support arrows for both this meaning in limits and the use of \( \rightarrow \) in function definitions. So users must get used to using equations rather than the standard math symbol for "approaches".

### 17.5 Finding minima and maxima by using Maple as a calculus calculator

A standard topic in elementary calculus is how to find the minimum or maximum of a continuous function in an interval (sometimes referred to as the \textit{local extrema of the function}) through the use of derivatives and a little algebra. The function that you are doing this to is sometimes referred to as the \textit{objective function}. Maple can be used to do the same calculations as you would do by hand -- to find the extrema of the objective function, take its derivative with respect to \( x \) (using \text{diff}), and then find where the derivative is equal to zero using \text{solve} or \text{fsolve}. The advantages of doing it with a system such as Maple are the same as with other calculations: a) it is easier to do correctly if the formula is involved or if you don't remember all the math, b) you can easily organize the information about the problem and its solution using Maple documents so that it can be easier to recall and presented, c) the speed of re-execution makes it easy to handle a bundle of similar extrema problems.

**Finding the minimum and maximum using textually-specified commands**

<table>
<thead>
<tr>
<th>From Anton, Calculus 8th ed. p. 307 problem 13 (modified)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Find the absolute maximum and minimum values of</td>
</tr>
<tr>
<td>( f := (x) \rightarrow 2 - 2 \cdot \sin(x) ) for ( x ) in ( -\frac{\pi}{4}, \frac{\pi}{4} )</td>
</tr>
<tr>
<td>( f := (x) \rightarrow 2 - 2 \cdot \sin(x) ) for ( x ) in ( -\frac{\pi}{4}, \frac{\pi}{4} )</td>
</tr>
<tr>
<td>Solution</td>
</tr>
<tr>
<td>( f := (x) \rightarrow 2 - 2 \cdot \sin(x) )</td>
</tr>
<tr>
<td>( x \rightarrow 2 - 2 \sin(x) )</td>
</tr>
<tr>
<td>We define a continuous function</td>
</tr>
<tr>
<td>And compute its derivative symbolically.</td>
</tr>
<tr>
<td>( \text{derive} := \text{diff}(f(x),x) )</td>
</tr>
<tr>
<td>( -2 \cos(x) )</td>
</tr>
<tr>
<td>We generate a plot to get an idea of where to look for minima and maxima. Evidently the minimum is in the interior of the interval and the maximum is at the left.</td>
</tr>
<tr>
<td>( \text{plot}(f,-\frac{\pi}{4},\frac{\pi}{4}) )</td>
</tr>
</tbody>
</table>
\[ x_{\text{min}} := \text{solve}(\text{deriv} = 0, x) \]

\[ \frac{1}{2} \pi \quad (17.24) \]

Finding where the derivative is zero should produce the minimum.

\[ f(x_{\text{min}}) \]

\[ 0 \quad (17.25) \]

The minimum is confirmed to be zero -- the graph seems to indicate that but the picture doesn't let us know whether it's really zero or just really small.

\[ f\left(-\frac{\pi}{4}\right) \]

\[ 2 + \sqrt{2} \quad (17.26) \]

Standard calculus procedure says to check out the value of the end points for the maximum. One is clearly larger than the other, and is bigger than the place within where the derivative is zero.

\[ f(\pi) \]

\[ 2 \quad (17.27) \]

Thus we conclude that \( f \) attains a maximum at \( x = -\frac{\pi}{4} \). The value there is \( 2 + \sqrt{2} \). The minimum is zero, at \( x = \frac{\pi}{2} \).

**Finding the minimum and maximum using textually-specified commands, part 2**

**Finding the** minimum and maximum using textually-specified commands, part 2

Anton, Calculus, 8th ed. Problem 307 (modified)

Find the absolute minimum of \( \left(x^2 + x + 5\right)^{\frac{2}{3}} \) for \( x \) in \([-2..3]\)

\( \left(x^2 + x + 5\right)^{\frac{2}{3}} \rightarrow \)

We enter the expression, and plot it on -2..3 using the 2-D plot builder.

We do a chain of calculations to find what is obviously the minimum point.
Compare to example 12.1.2 where because we assigned results to variables, we could just type in the name of the variable again. Here we have to copy the expression.

We had to copy the expression again, and type in "-1/2" to calculate the minimum value. Once we have a number, we can also see approximately what it is.

\[ (x^2 + x + 5)^{\frac{2}{3}} \] differentiate w.r.t \( x \)
\[ \frac{2}{3} \left(\frac{2x + 1}{(x^2 + x + 5)^{\frac{1}{3}}} \right) \]
solve for \( x \)
\[ \left[ x = -\frac{1}{2} \right] \]
\[ (x^2 + x + 5)^{\frac{2}{3}} \] evaluate at point
\[ \frac{1}{4} \]
\[ 19^{2/3} 4^{1/3} \]
at 5 digits
\[ 2.8258 \]

We turn the sequence of actions into a textual script. The parameters are the expression to be minimized, the variable in the expression and the region. The plot is just to help us have a visual check that the numeric answer result is probably correct. We could get Maple to tell us what the variable was, automatically, but that may be more work than we want to do.
\[ \text{deriv} := \text{diff}(\text{expr}, \text{var}) \]
\[ \frac{2}{3} \frac{2x + 1}{(x^2 + x + 5)^{1/3}} \]  

(17.31)

\[ \text{soln} := \text{solve}(\text{deriv}, x) \]
\[ \frac{1}{2} \]  

(17.32)

\[ \text{valAtPoint} := \text{eval}(\text{expr}, \text{var} = \text{soln}) \]
\[ \frac{1}{4} 192^{2/3} 4^{1/3} \]  

(17.33)

\[ \text{result} := \text{evalf}(\text{valAtPoint}) \]
\[ 2.825719660 \]  

(17.34)

We turn this into a textual script that does the same thing. Unless we print the plot, it doesn't appear because by default Maple only displays the result of the procedure. print causes something to be printed in addition to the final result.

```plaintext
defindMin := proc(expr, var, interval)
local deriv, soln, valAtPoint;
print(plot(expr, var=interval));
deriv := diff(expr, var);
soln := solve(deriv, var);
valAtPoint := eval(expr, var=soln);
return evalf(valAtPoint);
end;
end proc
```
We solve the minimization problem.

\[
\text{findMin}\left(\left(x^2 + x + 5\right)^{\frac{2}{3}}, x, -2.3\right)
\]

2.825719660 \hspace{1cm} (17.35)

Now that we have a procedure, we can solve a similar problem: minimize \(\sqrt{x^2 + x + 3}\) for \(y\) in the interval -1..4. We see that the plot supports the numerical calculation that the value of the expression at the minimum is approximately 1.65.
17.6 One-step extrema finding using maximize and minimize

Maple also provides both exact and approximate numerical functions that do the entire sequence of steps needed to find extrema. In this section, we discuss the Maple functions that provide \textit{exact} solutions to such problems. The Maple function for finding maxima is, appropriately enough, called \texttt{maximize}. If \texttt{expr} is an expression involving a variable \texttt{x}, then \texttt{maximize(expr,x=range)} will produce maximum value of the expression for \texttt{x} in that range. If there is no maximum value, or if Maple can't find it, NULL is returned. Thus if \texttt{maximize} returns an answer, it should be "believable", but the absence of an answer doesn't meant that there is no answer. \texttt{infinity} or \texttt{-infinity} may be an answer if the function is unbounded within the range.

We have already seen \texttt{max} and \texttt{min}, which are also built-in Maple functions. However, they only work on lists, sets of values. They don't work at finding minima or maxima of expressions.

It is often a good idea when working on extrema problems to do a plot of the expression in question, so that you can some notion of where the maxima will be and what their values will be like.

### minimize and maximize

\begin{quote}
From Anton, Calculus 8th ed. p. 307 problem 13 (modified)

Find the absolute maximum and minimum values of

\[ f := (x) \to 2 - 2\sin(x) \quad \text{for} \quad x \in \left[ \frac{\pi}{4}, \pi \right] \]

\textbf{Solution}

\[ f := (x) \to 2 - 2\sin(x) \]

\[ x \mapsto 2 - 2\sin(x) \quad (17.37) \]

We define a function.

\[ \text{plot}\left(f, \left[ \frac{\pi}{4}, \pi \right]\right) \]

We generate a plot to get an idea of where to look for minima and maxima. Evidently the minimum is in the middle and the maximum is at the left.

\[ \text{smallestValue} := \text{minimize}\left(f(x), x = \left[ \frac{\pi}{4}, \pi \right]\right) \]

\[ 0 \quad (17.38) \]

The minimum is confirmed to be zero -- the graph seems to indicate that but the picture doesn't let us know whether it's really zero or just really small. Note again that this is the minimum value, it is not the location of the minimum value.
\[\text{largestValue} := \text{maximize}\left( f(x), x = -\frac{\pi}{4}, \pi \right) \]
\[2 + \sqrt{2}\]  \hspace{1cm} \text{Finding the maximum is a copy/edit job once we've worked out how to do the minimum.}  \hspace{1cm} (17.39)

Sometimes, you want not only what the maximum or minimum value is, but also \textit{where} it is. If you give \texttt{maximize} or \texttt{minimize} an extra third parameter, location, then it will return a sequence for a result. This sequence has a logical but somewhat intricate structure. You can use a chain of selections to extract the location \texttt{var= location point} from the answer given by Maple.

**Solving a problem with maximize and minimize**

From Anton, Calculus 8th ed. p. 307 problem 13 (modified)
Find the absolute maximum and minimum values of
\[f := (x) \rightarrow 2 - 2 \cdot \sin(x) \text{ for } x \text{ in } -\frac{\pi}{4}, \pi\]

Solution
\[f := (x) \rightarrow 2 - 2 \cdot \sin(x)\]
\[x \rightarrow 2 - 2 \sin(x)\]  \hspace{1cm} \text{We define a function.}  \hspace{1cm} (17.40)

\[\text{plot}\left( f, -\frac{\pi}{4}, \pi \right)\]  \hspace{1cm} \text{We generate a plot to get an idea of where to look for minima and maxima. Evidently the minimum is in the middle and the maximum is at the left.}

\[\text{smallestValue} := \text{minimize}\left( f(x), x = -\frac{\pi}{4}, \pi \right)\]
\[0\]  \hspace{1cm} \text{This is what we did in Example 12.2.1 -- we just get the smallest value of } f.  \hspace{1cm} (17.41)

\[\text{minResult} := \text{minimize}\left( f(x), x = -\frac{\pi}{4}, \pi, \text{location} \right)\]
\[0, \left\{ x = \frac{1}{2}, \pi \right\}, 0 \right\}\]  \hspace{1cm} \text{This is what minimize does with the extra parameter location.}  \hspace{1cm} (17.42)

\[\text{minLocation} := \text{rhs}(\text{minResult}[2][1][1][1])\]  \hspace{1cm} \text{This longwinded sequence of selections is needed to get just the value of the minimum. We need to extract the second part of the}
If you want to evaluate another expression at a minimum point, then eval(expr, result[2][1][1]) will do that. This is because result[2] contains a list which describes the location/value pair. result[2][1] returns the first element of that, which is a list containing a particular location/value pair. result[2][1][1] is a set of the form \{var = extreme value\}. eval(expr, \{var=value\}) will evaluate expr with value replacing all occurrences of var in expr.

Optimization in a production problem with minimize and maximize


A liquid form of penicillin manufactured by a pharmaceutical firm is sold in bulk at a price of $200 per unit. If the total production cost (in dollars) for \( x \) units is

\[ C(x) = 500000 + 80x + 0.003x^2 \]

and if the production capacity of the firm is at most 30,000 units in a specified time, how many units of penicillin must be manufactured and sold in that time to maximize the profit? What will the revenue be at the maximal profit?

Solution

\[
\begin{align*}
C & := (x \rightarrow 500000 + 80x + 0.003x^2) \\
R & := (x \rightarrow 200x) \\
P & := (x \rightarrow R(x) - C(x)) \\
\end{align*}
\]

We define a Cost, Revenue, and Profit function, using the information given in the problem about cost, and the standard definitions from economics for revenue and profit. We could use the proc..end proc way of defining a function to equal effect but the arrow notation was designed to make it convenient to enter these short definitions that don't need if or for.

We plot the profit function to scope out its behavior. Evidently the function does hit a maximum, roughly around \( x=20000 \). If we got \( x=35 \) as the answer from our maximum calculation in Maple, we would wonder if we did things correctly.
\[
\text{maxResult := maximize}(P(x), x = 0 \ldots 30000, \text{location})
\]
\[
\begin{aligned}
7.00000 \times 10^5, & \{ \{x = 20000\} , 7.00000 \times 10^5 \} \\
\text{maxLocation := floor}(\text{rhs}(\text{maxResult}[2][1][1]))
\end{aligned}
\]
\[
\begin{aligned}
&\quad \quad 20000 \\
\text{maxProfit := maxResult}[1]
\end{aligned}
\]
\[
\begin{aligned}
&\quad \quad $700,000.00 \\
\text{revenueAtMax := R}(\text{maxLocation})
\end{aligned}
\]
\[
\begin{aligned}
&\quad \quad $4,000,000.00 \\
\text{costAtMax := C}(\text{maxLocation})
\end{aligned}
\]
\[
\begin{aligned}
&\quad \quad $3,300,000.00
\end{aligned}
\]

We are getting floating point answers from Maple because some of the numbers in C are floating point. If we wanted an exact calculation, we would change the .003 into 3/1000. We use the daisy-chaining of operations we saw in the previous example to extra the value of the \(x\) where the maximum is obtained. We use the \text{floor} function which rounds down to the nearest integer, since we can't produce a fraction of a unit. We used the clickable menu item "Numeric Formatting" to reformat the result in terms of currency.

\section*{17.7 One-step approximations to extrema using Optimization[Maximize] and Optimization[Minimize]}

Maple has solve and fsolve that provide exact and approximate solutions. Similarly there is Optimization[Maximize] and Optimization[Maximize] play the role of fsolve to maximize and minimize. These functions only work when there is a numerical answer, and then only try to produce a "best effort" approximation which despite "best effort" of the programmers may not necessarily be that good an approximation. Its quality should always be tested against expectations, and to see how well it works. The clickable interface's "Optimization" operation also invokes the approximate optimizers.

\text{Optimization[Maximize]} and \text{Optimization[Minimize]} always return a list. The first item of the list is the extreme value, and the second item in the list is a list of equations describing the location of the extreme value. As with the exact optimizing functions \text{maximize} and \text{minimize}, the second item can be used as a parameter to eval to evaluate an expression at the location of an extreme value.

\textbf{Finding an approximation to the minimum with Optimization[Minimize]}

Anton, Calculus 8th ed. problem 47, p. 321 (modified)
Two particles, A and B, are in motion in the xy-plane. Their coordinates at each instant of time \( t (t \geq 0) \) are given by \( x_A = t \), \( y_A = 2t \), \( x_B = 1 - t \), \( y_B = t \). Find the minimum distance between A and B.

\[
\begin{align*}
x_A &:= t \\
y_A &:= 2t \\
x_B &:= 1 - t \\
y_B &:= t
\end{align*}
\]

(17.55) (17.56) (17.57) (17.58)

\[
p1 := [x_A, y_A] \\
p2 := [x_B, y_B]
\]

(17.59) (17.60)

\[
dist(p1, p2) = \sqrt{5t^2 - 4t + 1}
\]

(17.62)

\( p1 \) and \( p2 \) are lists of the x and y positions for each point. \( dist \) is the objective function. This is the Euclidean distance between the two points. We can compute the distance between \( p1 \) and \( p2 \) using it.

It's always good to have an independent way of evaluating the answer you get from a "solver". In this case, we can plot the
function and from that see approximately where the answer ought to be. Our first plot does not show the minimum that clearly, so we do another plot that emphasizes the region of interest.

\[
\text{plot}(\text{dist}(p1, p2), t = 0.2)
\]

\[
\text{with(Optimization)}
\]

\[
\text{[ImportMPS, Interactive, LPsolve, LSSolve, Maximize, Minimize, NLPSolve, QPSolve]}
\]

\[
\text{minResult := Minimize(dist(p1, p2), t = 0.2)}
\]

\[
[0.44721359549958150, [t = 0.400000000000000222]]
\]

\[
\text{minLocation := \text{minResult}[2]}
\]

\[
[t = 0.400000000000000222]
\]

\[
\text{eval}(p1, \text{minLocation})
\]

\[
[0.400000000000000222, 0.800000000000000]
\]

\[
\text{eval}(p2, \text{minLocation})
\]

\[
\]

Using \text{with}, we can then say \text{Minimize} rather than Optimization[Minimize].

Note that the exact location is \( t = 0.4 \), but that rounding error made during the operation of Minimize leads to only an approximation of the exact answer. This is typical.

According to the structure returned by this function, the location information is in the second element of the result.

We can find the location of particles A and B at \( t = 0.4 \) by evaluating the expression \( p1 \) and \( p2 \) at that value of \( t \). Note that \text{minLocation} is a list consisting of an equation, which \text{eval} can use as its second argument.
\[
\begin{array}{l}
\text{minimize}(\text{dist}(p1, p2), t = 0.2) \\
\frac{1}{5} \sqrt{3} \\
\text{at 10 digits} \\
0.4472135954
\end{array}
\]

Note that this is a problem that Maple's exact solver can solve exactly. The numerical approximation produced by Optimization[Minimize] is in close agreement with the exact result.

\[
T := t^2 + 5 \cdot 7^2 - 35 \cdot t + 3
\]

\[
\frac{t^2 + 5 \cdot 7^2 - 35 \cdot t + 3}{(17.70)}
\]

This expression has a complicated expression for the exact value. We could try to approximate it using the right-click->approximate menu, or we could give it to the approximate minimizer instead.

\[
\text{plot}(T, t = 0.2)
\]

![Graph of T](image)

\[
\text{minimize}(T, t = 0.2) \\
3 + \text{RootOf}(7 \cdot Z^2 + 10 \cdot Z - 35, \text{index} = 1)^2
\]

\[
\frac{3 + \text{RootOf}(7 \cdot Z^2 + 10 \cdot Z - 35, \text{index} = 1)^2}{(17.71)}
\]

\[
\text{at 5 digits} \\
-28.236
\]

Optimization[Minimize](T, t = 0.2)

\[
[-28.2356100235686540, \{t = 1.21771410298014682\}]
\]

(17.73)
17.8 Multivariate operations in Maple

`diff`, `minimize`, `solve`, and the rest can handle problems and situations where there are multiple variables involved. Approximation methods (e.g. `fsolve` or Optimization[Maximize] often take more prominence with multivariate problems because of the difficulty and/or expense of finding of succinct formulas that express the solution. We won't explore Maple's abilities to handle multivariate problems here, but they are there ready to be explored and used when you're ready for it.

### Multivariate operations in Maple

\[
\frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} \sin \left( \frac{x}{y} \right)^2 \right) = \frac{2 \cos \left( \frac{x}{y} \right)^2}{y^3} - \frac{2 \sin \left( \frac{x}{y} \right)^2}{y^3} \tag{17.74}
\]

Compute the second partial derivative with respect to \(x\).

\[
diff\left( \sin \left( \frac{x}{y} \right)^2, x, y \right) = \frac{2 \cos \left( \frac{x}{y} \right)^2}{y^3} \left( \frac{x}{y} \right) - \frac{2 \sin \left( \frac{x}{y} \right)^2}{y^3} \left( \frac{x}{y} \right) - \frac{2 \sin \left( \frac{x}{y} \right) \cos \left( \frac{x}{y} \right)}{y^2} \tag{17.75}
\]

The textual way of entering partial derivatives.

\[
eqns := \begin{cases} x + 2y = \frac{5}{4}, & x^2 + y^2 = 1 \\ x + 2y = \frac{5}{4}, & x^2 + y^2 = 1 \end{cases} \tag{17.76}
\]

\[\eqnsf := \begin{cases} x + 2y = 1.25, & x^2 + y^2 = 1 \\ x + 2y = 1.25, & x^2 + y^2 = 1 \end{cases} \tag{17.78}
\]

We are interested in finding the points of intersection with the unit circle with center at (0,0), and the line defined by \(x+y=5\). Doing it exactly gives an answer, but it's not easy to understand -- \(x\) and \(y\) can be either the positive or negative solution to a quadratic expression. Introducing a floating point number into the equation (1.25 instead 10/8) tells `solve` that it's all right to approximate the results.

\[
\text{solve}(\eqns, \{x, y\}) = \begin{cases} x = -\frac{1}{2} \text{RootOf}(5 \cdot Z^2 - 20 \cdot Z + 9, \text{label} = _L1641) + \frac{5}{4}, \\ y = \frac{1}{4} \text{RootOf}(5 \cdot Z^2 - 20 \cdot Z + 9, \text{label} = _L1641) \end{cases} \tag{17.77}
\]

\[
\text{solve}(\eqnsf, \{x, y\}) = \begin{cases} x = 0.9916198487, y = 0.1291900756, \\ x = -0.9916198487, y = 0.8708099244 \end{cases} \tag{17.79}
\]

\[
expr := \sqrt{\left( x^2 + \sin \left( y + \frac{1}{10} \right)^2 \right)} + 5 \tag{17.80}
\]

\[
\sqrt{\left( x^2 + \sin \left( y + \frac{1}{10} \right)^2 \right)} + 5
\]

We want to find the minimum value of an expression in two variables. `minimize` takes a long time to work and doesn't seem to come up with much. However, the approximate minimizer finds a value of 5.

\[
\text{minimize}(\expr, x = -1 .. 1, y = -1 .. 1)
\]

Warning, computation interrupted

\[
\text{Optimization\{Minimize\}(\expr, x = -1 .. 1, y = -1 .. 1)
\]
### 17.9 Chapter summary

Symbolic differentiation using the expression palette, right-click, control-\textasciitilde.

\[
x^2 - 2x + 5 \quad \text{differentiate w.r.t. } x \rightarrow \ 2x - 2
\]

Enter the expression, and then right-click (control-click on Macintosh) to bring up the calculation options. Select differentiate-\textasciitilde x.

\[
\sin(\omega t + 5)^2 \quad \text{differentiate w.r.t. } \frac{\partial}{\partial \omega} \rightarrow 2 \sin(\omega t + 5) \cos(\omega t + 5) \omega
\]

We can calculate the second derivative with respect to \( t \) of this expression by performing differentiation twice. We simplify the expression a bit by performing the operation simplify-\textasciitilde symbolic.

\[
\frac{d}{d\omega} \sin(\omega t + 5)^2 = 2 \sin(\omega t + 5) \cos(\omega t + 5) \ t
\]

If we want the document to display the mathematical notation for the derivative, we can select \( \frac{d}{dx} f \) from the expression palette and then fill in the slots \( f \) and \( x \). We can then get Maple to calculate the derivative by typing control-\textasciitilde.

Symbolic differentiation and evaluation of derivatives, textually

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( expr := x^2 - 2x + 5 )</td>
<td>Find the first derivative of the expression with respect to ( x ).</td>
</tr>
<tr>
<td>( x^2 - 2x + 5 )</td>
<td>(17.82)</td>
</tr>
<tr>
<td>( \text{diff}(expr, x) )</td>
<td>Find the first derivative of the expression with respect to ( t ).</td>
</tr>
<tr>
<td>( 2x - 2 )</td>
<td>(17.83)</td>
</tr>
<tr>
<td>( posExpr := \sin(\omega t + 5)^2 )</td>
<td>Find the second derivative of the expression with respect to ( t ). The result is the same as if we had taken the derivative of 1.2.4.</td>
</tr>
</tbody>
</table>
\[
\sin(\omega t + 5)^2
\]
\[\text{diff}(\text{postExpr}, t)\]
\[
2 \sin(\omega t + 5) \cos(\omega t + 5) \omega
\]
\[\text{diff}(\text{postExpr}, t, t)\]
\[
2 \cos(\omega t + 5)^2 \omega^2 - 2 \sin(\omega t + 5)^2 \omega^2
\]
\[\text{diff}(\text{expr}, t)\]
\[
0
\]
\[\text{simplify}(1.9.5)\]
\[
2 \omega^2 \left(2 \cos(\omega t + 5)^2 - 1\right)
\]
\[\text{eval}(1.9.2, x = 3)\]
\[
4
\]
\[\text{eval}(1.9.7, t = 47.0)\]
\[
2 \omega^2 \left(2 \cos(47.0 \omega + 5)^2 - 1\right)
\]

### Textual version of limits

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{limit}(1/(x-3)^2, x=3))</td>
<td>The first argument is the expression that you wish to take the limit of. The second argument is an equation indicating the limit variable and the limiting value.</td>
</tr>
<tr>
<td>(\text{limit}(\sin(x)/x, x=\text{infinity}))</td>
<td>You can use &quot;infinity&quot; or &quot;-infinity&quot; as the way of specifying (\infty) or (-\infty) textually without use of the palette.</td>
</tr>
<tr>
<td>(\text{limit}(1/t, t=0, \text{left}))</td>
<td>The third optional third argument to (\text{limit}) can specify a one sided limit. This is the textual way of specifying (\lim_{t \to 0^-} \frac{1}{t}). There is no simple way of specifying this</td>
</tr>
<tr>
<td>(\text{limit}(1/t, t=0, \text{right}))</td>
<td>This is the textual way of specifying (\lim_{t \to 0^+} \frac{1}{t}).</td>
</tr>
</tbody>
</table>

### Minima and maxima

<table>
<thead>
<tr>
<th>Operation</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact minimum or maximum value</td>
<td>(f := (x) \to 2 - 2 \sin(x)) for (x) in ([-\frac{\pi}{4}, \pi])</td>
</tr>
<tr>
<td>Solution</td>
<td>(f := (x) \to 2 - 2 \sin(x)) (x \to 2 - 2 \sin(x))</td>
</tr>
</tbody>
</table>
It's usually a good idea to plot the objective function to get some sense of where the extrema are. Remember that the function to be investigated should be continuous if you expect good results from calculus techniques.

\[
\text{smallestValue} := \min(f(x), x = -\frac{\pi}{4}, \pi)
\]

\[0\]  \hspace{2cm} (17.96)

\[
\text{largestValue} := \max(f(x), x = -\frac{\pi}{4}, \pi)
\]

\[2 + \sqrt{2}\]  \hspace{2cm} (17.97)

**Exact minimum or maximum value location**

\[
\text{minResult} := \min(f(x), x = -\frac{\pi}{4}, \pi, \text{location})
\]

\[0, \{\{x = \frac{1}{2}\pi\}, 0\}\]  \hspace{2cm} (17.98)

\[
\text{maxResult} := \max(f(x), x = -\frac{\pi}{4}, \pi, \text{location})
\]

\[2 + \sqrt{2}, \{\{x = -\frac{1}{4}\pi\}, 2 + \sqrt{2}\}\]  \hspace{2cm} (17.99)

**Evaluation of an function at an extremum point**

\[
g := (t) \mapsto \sin(2t) \quad \text{Evaluate g at the location of the smallest value of f.}
\]

\[
t \mapsto \sin(2t)
\]

\[
\text{minLocation} := \mathrm{rhs}(\text{minResult}[2][1][1][1])
\]

\[
\]
\begin{align*}
g(\text{minLocation}) &= \frac{1}{2} \pi \\
\text{Evaluation of an expression at an extremum point} \quad &
\begin{align*}
\text{expr} &:= x^2 + \sin(x) \\
\text{minEvalpt} &:= \text{minResult}[2][1][1] \\
\quad &= \left\{ \begin{array}{c}
x = \frac{1}{2} \pi \\
\end{array} \right.
\end{align*} \\
\text{eval}(\text{expr}, \text{minEvalpt}) &= \frac{1}{4} \pi^2 + 1
\end{align*}
\begin{align*}
\text{Approximate minimum or maximum value and location} \\
\text{approxMinResult} &:= \text{Optimization}[\text{Minimize}]
\left( f(x), x = \frac{\pi}{4} \right) \\
\quad &= [0, x = 1.57079632679489478] \\
\text{approxMaxResult} &:= \text{Optimization}[\text{Maximize}]
\left( f(x), x = \frac{\pi}{4} \right) \\
\quad &= [3.1415926535897932385, x = -0.785398151694103208] \\
g(\text{rhs(approxMaxResult}[2][1])) &= -1.000000000 \\
\text{eval}(\text{expr}, \text{approxMaxResult}[2]) &= -0.0902565162
\end{align*}
18 More support for mathematical modeling

18.1 Chapter Overview

We introduce three more built-in computational features that Maple has:

Solving integration problems (computing integrals)

Piecewise expressions

Curve fitting using splines

Mathematical models of many kinds of scientific or engineering situations often lead the investigator to pose problems that can be solved through these features.

18.2 Recalling basic facts about integration

We review some of the basic facts and terminology about integration from your calculus course.

Definite can be related to finding the area under a curve described by a function. The definite integral \( \int_a^b f(x) \, dx \)

is notation for this. If an answer for a definite integration problem exists, then it is a number which could be described exactly or as an approximation.

Indefinite integration is sometimes called the problem of antidifferentiation. That is, given a function \( f \), we must find another function \( F \) whose derivative is \( f \):

\[ \frac{d}{dx} F(x) = f(x) \, . \]

Another way of writing this relationship is:

\[ \int f(x) \, dx = F(x) \, . \]

In most cases, there is no single function \( F(x) \) that solves the antiderivative problem for \( f(x) \). There is a whole family of functions that differ from each other by a constant. Sometimes the antiderivative \( F(x) \) is written with an extra symbolic constant, as in \( \int f(x) \, dx = F(x) + C \), particularly if antidifferentiation is going to be applied as only one step of a problem-solving process. Since the hard part of doing indefinite integration is finding \( F(x) \) without the constant and since the name of the constant can be freely chosen, systems that can compute antiderivatives often omit the constant in their answers, letting the user add in the constant of their choice later if they need to.

The answer \( F(x) \) is expected to be a formula involving expressions and functions that the user already knows about. \( F(x) \), when it exists, is referred to as a closed form solution to the integration problem. For calculus freshmen, these are expressions that involve numbers, polynomials, roots, rational functions, trig, logarithm and exponential functions. For more advanced mathematics students, it might include other functions: \( \text{erf} \) (error function), \( Ei \) (exponential integral), Bessel, \( \Gamma \), etc.

The Fundamental Theorem of Calculus (FTC) states that we can evaluate a definite integral in terms of an antiderivative via the formula
\[ \int_{a}^{b} f(x) \, dx = F(b) - F(a), \]

where \( F \) is any function such that \( \frac{d}{dx} F(x) = f(x) \).

Programs for doing indefinite integration/antidifferentiation have been around from the early '60s. They can be found in a number of symbolic systems: Maple, Mathematica, Macsyma (now found in SAGE), and MuPad (now part of Matlab). It is not an easy program to write. Even today in the era of "open source", the quality and power of such programs varies significantly.

The ideas for how to program an indefinite integrator comes from seminal work done by Robert Risch in the late 1960s, who was in turn continuing work done previously by other mathematicians such as Joseph Liouville (1809-1882). Earlier approaches in the early '60s treated symbolic anti-differentiation as a problem in artificial intelligence. Those AI programs (such as SAINT by James Slagle (1934-1994) and SIN by Joel Moses (1941-) tried to simulate the expertise of a college calculus student. However, they would sometimes give up on problems that could be solved through a clever trick of substitution or other reformulation. Risch's work used advanced mathematics to come up with programs that would guarantee that they could find a solution if there was one to be found, for certain classes of integrands. However, while programs produce an answer, it is not necessarily in the same form as given by a mathematics textbook although it would be equivalent or differ only by a constant.

### 18.3 Indefinite integration

#### Doing anti-differentiation with int

<table>
<thead>
<tr>
<th>Indefinite integration with int</th>
</tr>
</thead>
<tbody>
<tr>
<td>The clickable interface can be used to input an indefinite integration problem for Maple to compute. Use the ( \int ) ( dx ) item in the Expression Palette, then edit the expression ( f ) and the variable of integration ( x ).</td>
</tr>
<tr>
<td>The general form of the textual version of the integration command is</td>
</tr>
<tr>
<td>( \text{int} ( expression , , , var ) )</td>
</tr>
<tr>
<td>This computes the antiderivative of the expression, with respect to the variable ( var ).</td>
</tr>
</tbody>
</table>

#### Indefinite integration with int

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \int_{x}^{x} f(x) , dx ]</td>
<td>The clickable interface can be used to enter the integration problem.</td>
</tr>
<tr>
<td>[ \int \frac{1}{x} , dx ] ( \ln(x) )</td>
<td>Note that in the clickable interface there are two ways of entering what is &quot;( \cos(x) ) times ( \cos(x) )&quot;.</td>
</tr>
<tr>
<td>[ \int \cos(x)^2 , dx ] ( \frac{1}{2} \cos(x) \sin(x) + \frac{1}{2} x ) (18.2)</td>
<td></td>
</tr>
</tbody>
</table>
\[
\frac{1}{2} \cos(x) \sin(x) + \frac{1}{2} x
\]  

(18.3)

\[
\text{int}(\sqrt{x}, x)
\]

\[
\frac{2}{3} x^{3/2}
\]  

(18.4)

\[
\text{int}\left( \cos(\alpha) \cdot \sin\left(\frac{\alpha}{2}\right), \alpha \right)
\]

\[
-\frac{1}{3} \cos\left(\frac{3}{2} \alpha\right) + \cos\left(\frac{1}{2} \alpha\right)
\]  

(18.5)

| expr := \tan(t) + x^2;  
| il := int(expr, t);  
| dl := diff(il, t);  
| difference := dl-expr;  
| print("difference":. difference);  
| print("simplified difference":, simplify(difference));  

| \tan(t) + x^2  
| -\ln(\cos(t)) + x^2 t  
| \frac{\sin(t)}{\cos(t)} + x^2  
| \frac{\sin(t)}{\cos(t)} - \tan(t)

"difference": \left( \frac{\sin(t)}{\cos(t)} - \tan(t) \right)

"simplified difference": 0

Understanding the answers from \text{int}

If Maple does not know how to find an answer it will return an expression that is the display version of the input expression. It does this instead of giving an error message or NULL so that a program using integration can continue even if the attempt to integrate fails. For example, even if you don't have the anti-derivative of the integrand, you might want to do approximate definite integration with it (see \(<<\text{Brian: add section link to section on definite integration}>>\)).

Sometimes the answer given by the symbolic integrators will be given in terms of functions that beginning students haven't heard of. This can be confusing, but most systems do not have a "beginners mode" that avoids using hard math. If you see a function you don't know of, you can get a brief explanation of it through on-line help. Whether or not the answer in this form is satisfactory for you depends on the situation. If it's for classroom use, and they want you to find answers that have only "freshman calculus functions", e.g. trig functions, logarithms, polynomials, exponential functions, then the answer as given will not be of much help. It might be a sign that you asked \text{int} to solve something different than what the problem intended, or it might mean that there's a way to simplify the strange-looking answer into something you recognize. If it's a situation on the job, it may be worthwhile to follow the answer as far as you need to in order to get the job done, even if it goes beyond the math you already know.
To summarize, the responsibilities that users of symbolic integration programs have are a) to be prepared to learn about more advanced functions if answer is of that form and they need to work with it, and b) understand when the computer system is telling them that it can't find a closed form solution.

### Integration answers using advanced functions

#### Integration answers using advanced functions

\[
\int e^x \, dx = \frac{1}{2} \sqrt{\pi} \text{erf}(x)
\]

(18.6)

You wouldn't see this integration problem in an elementary calculus textbook, because the answer involves the "error function". The term "error function" is not indication that there is a mistake, it's the official terminology for the function, in the same way that "sine", "cotangent" or "exponential" are names of functions. The name arose from the function's original use in probability and statistics. You can read more about the error function by typing "erf" or "error function" into Maple's on-line help.

\[
\int \left( \frac{1}{\log(r)^2} \cdot r \right) \, dr = -Ei(1, -\ln(r))
\]

(18.7)

The "Ei" function is referred to as the "exponential integral function".

\[
\int \frac{(x^3 + x^2 + 1)}{(x-1)(x^2 + 2x + 1)} \, dx
\]

This a situation where there is a closed form solution, but it involves the roots of a polynomial that cannot be expressed in terms of conventional square roots, cube roots, \( n \)th roots, etc.

\[
R = \text{RootOf}(11137 \cdot \hat{z}^2 - 11137 \cdot \hat{z}^4 + 9600 \cdot \hat{z}^3 - 9504 \cdot \hat{z}^2 - 17984 \cdot \hat{z} + 32512)
\]

\[
\ln x + \frac{349074238374535}{88857035100224} \cdot R^4
\]

\[
+ \frac{121804780837461}{88857035100224} \cdot R^3 + \frac{2829477036907}{22214258775056} \cdot R^2
\]

\[
- \frac{7796211858445}{13883911798441} \cdot R - \frac{8123093952608}{13883911798441}
\]

\[
+ \frac{3}{4} \ln(x-1)
\]

This answer includes a sum over all the zeroes \( R \) (complex or real) of the polynomial equation

\[
1317 \cdot \hat{z}^5 - 11137 \cdot \hat{z}^4 + 9600 \cdot \hat{z}^3 - 9504 \cdot \hat{z}^2 - 17984 \cdot \hat{z} + 32512
\]

This situation (complex) says that these roots have the approximate values

\[
-1.060190349, -0.06877225771 - 1.3885265691, -1.3885265691, 1.09867432 + 0.44104008751, 1.09867432 + 0.44104008751
\]

This is one of those situations where it will be difficult to get much use out of the symbolic answer. If you are really doing a definite integration problem involving this integrand, pursuing an approximation to the definite integral as discussed in Section 17.2, may be a better ploy.

### Integration answers when Maple can't find an expression for the anti-derivative

\[
\text{answer} := \text{int}(f(x), x)
\]

(18.9)

Maple can't find the antiderivative because we haven't told it what the function \( f \) is.

It can't find the symbolic integral for \( g \) either, for the same reason. We see that \( \text{printf} \) says that the value of \( \text{answer2} \) is, in textual form, the same as the input was. Expression 1,2,3,4 would print out the same way if we used print rather than Maple's default 2d math output to display that result.
\( \int \frac{1}{\sqrt{\ln(\sin(x) + 1 + e^x)}} \, dx \) (18.10) Here's another example of an integral that Maple can't do even though the expression being integrated is something that it recognizes.

### Checking that your answer is correct

**Is the derivative of your answer the same or equivalent to what you started with?**

It is no better to trust blindly the answer you get from \textit{int} than anything else you calculate with a computer or calculator. You could have mis-entered the integrand or entered a garbled form of the \textit{int} command, producing a "garbage in, garbage out" situation. Fortunately because of the relationship between integration and differentiation, it is possible to check integration answers without having to do the integration by hand:

1. Use \textit{int} to calculate the answer

1. Apply \textit{diff} to the answer, to see (by visual inspection) if you end up with the integrand (the original function you were trying to integrate).

1. If the derivative does not look the same as the integrand, try \textit{simplify} on the difference between the result of step 2 and the original expression. If you get zero, the two are equivalent.

### Checking answers

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{expr} := 1 + x + \frac{\cos(x)}{2} ) ( \begin{align*} 1 + x + \frac{1}{2} \cos(x) \end{align*} ) (18.11)</td>
<td>The derivative of the symbolic integral is the same as the original expression. This is obvious just by &quot;eyeballing&quot; it, but we can also confirm that the difference between \textit{expr} and the derivative is 0.</td>
</tr>
<tr>
<td>( \text{answer} := \text{int} (\text{expr}, x) ) ( x + \frac{1}{2} x^2 + \frac{1}{2} \sin(x) ) (18.12)</td>
<td></td>
</tr>
<tr>
<td>( \text{dAnswer} := \text{diff} (\text{answer}, x) ) ( 1 + x + \frac{1}{2} \cos(x) ) (18.13)</td>
<td></td>
</tr>
<tr>
<td>( \text{expr} - \text{dAnswer} ) ( 0 ) (18.14)</td>
<td></td>
</tr>
<tr>
<td>( \text{expr} := \sin(x)^2 ) ( \sin(x)^2 ) (18.15)</td>
<td>The derivative of the symbolic integral does not appear at first glance to be the same as the original expression. However, the difference between them does simplify to zero, so they are equivalent. Maple does not automatically employ trig identities such as ( \sin^2(a) + \cos^2(a) = 1 ) because it would often introduce tedious delay to try to apply all possible identities all the time.</td>
</tr>
<tr>
<td>( \text{answer} := \text{int} (\text{expr}, x) ) ( -\frac{1}{2} \cos(x) \sin(x) + \frac{1}{2} x ) (18.16)</td>
<td></td>
</tr>
<tr>
<td>( \text{dAnswer} := \text{diff} (\text{answer}, x) )</td>
<td></td>
</tr>
</tbody>
</table>
\[
\frac{1}{2} \sin(x)^2 - \frac{1}{2} \cos(x)^2 + \frac{1}{2} \\
\]

\[
d := expr - dAnswer \\
\frac{1}{2} \sin(x)^2 + \frac{1}{2} \cos(x)^2 - \frac{1}{2} \\
\]

\[
simplify(d) \\
0 \\
\]

\[
expr := (x^2 + 1)/(x-1) \\
\frac{x^2 + 1}{x - 1} \\
\]

\[
answer := \text{int}(expr, x) \\
\frac{1}{2} x^2 + x + 2 \ln(x - 1) \\
\]

\[
dAnswer := \text{diff}(answer, x) \\
x + 1 + \frac{2}{x - 1} \\
\]

\[
simplify(expr - dAnswer) \\
0 \\
\]

Here is another example of a symbolic integration answer whose derivative does not appear to be the same as the starting expression. However, simplification of the difference between the two indicates that the two are equivalent because their difference simplifies to zero.

Sometimes the "simplify the difference to zero" trick is not needed to check work because it is obvious that the two are the same. Maple has other simplification operations such as normal, factor, convert, combine, and expand that will attempt to put results into alternative forms. See on-line help about simplify and the other operations for more information.

**Troubleshooting int for indefinite integration problems**

`int` is intended to work with expressions involving the variable of integration, not names of functions. If an expression does not seem to involve the variable of integration, it is treated as a symbolic constant, regardless of whether it is also the name of a function. This can lead to misleading "answers".

**Variablesexpressions**

| Variables assigned expressions work with int, but not variables assigned function definitions |
|----------------------------------------|------------------------------------------------------------------------------------------------|
| **badResult** := \[ \int \cos dt \] | Since the integrand is just \( \cos \), and does not involve the variable of integration, Maple treats it as a symbolic constant. Since for a name \( a \), \[ \int a\,dt = at \], the result of this integral is the name "cos", times \( t \). It sort of looks like \( \cos(t) \) but it isn't. |
| eval(badResult, \( t = 0 \)) | One way to demonstrate that this answer isn't \( \cos(t) \) is to evaluate it at \( t=0 \). \( 0 \cdot \cos = 0 \). That certainly isn't the value of \( \cos(0) \). |
Variables assigned expressions work with \texttt{int}, but not variables assigned function definitions

\begin{align*}
\text{eval(badResult, } t = .5) & \\
& 0.5 \cos t \\
\text{(18.26)} & \\
\text{goodResult := } & \int \cos(t) \, dt \\
& \sin(t) \\
\text{(18.27)} & \\
\text{eval(goodResult, } t = 0) & 0 \\
\text{(18.28)} & \\
\text{eval(goodResult, } t = .5) & 0.479425386 \\
\text{(18.29)} & \\
g := x \rightarrow x^2 & \\
& x \rightarrow x^2 \\
\text{(18.30)} & \\
\text{int}(g, x) & gx \\
\text{(18.31)} & \\
\text{int}(g(x), x) & \frac{1}{3} x^3 \\
\text{(18.32)} & \\
\text{int}(g(t), t) & \frac{1}{3} t^3 \\
\text{(18.33)} & \\
\text{int}(g(y), x) & y^2 x \\
\text{(18.34)} & \\
\text{int(abs(b), b)} & \\
& \begin{cases} \\
-\frac{1}{2} b^2 & b \leq 0 \\
\frac{1}{2} b^2 & 0 < b \\
\end{cases} \\
\text{(18.35)} & \\
\text{procAbs := proc(x)} & \\
& \text{if } x \geq 0 \text{ then} \\
& \text{return } x \\
& \text{else} \\
& \text{return } -x \\
& \text{end if;} \\
& \text{end} & \\
\end{align*}

Evaluating this expression for another value indicates that Maple's integration result is actually "\(t^*\cos\)"; the variable \(t\) times the name of a function.

Giving the symbolic integrator an expression in \(t\), gives a proper result.

Another mistaken attempt to integrate a function name instead of an expression involving the integration variable. Like the example above, it produces the answer "\(g\) times \(x\). Note that the value of \(g\) is not the expression \(x^2\), it's the function definition \(x \rightarrow x^2\).

Evaluating \(g(x)\) does produce \(x^2\), which \texttt{int} can integrate properly.

Evaluate \(g(t)\) produces an expression in \(t\), which \texttt{int} can integrate properly if it is given that as the variable of integration.

This is the integral of \(y^2\) with respect to \(x \rightarrow x\) times \(y^2\).

Integrating the absolute value function returns a piecewise expression.

We write a Maple procedure to emulate the absolute value function...

But it doesn't work. This is because the if statement gives an error when it tries to decide whether \(x\) is greater than zero when there is no numerical value for \(x\).

We are not saying that you can't use a function defined through a Maple procedure in symbolic integration. If the procedure can execute and return a symbolic expression even if its arguments are symbols, then there is no problem.

This computes the integral of \(\exp(y+1)*(y+1)\). Note that we could use it to calculate the integral of \(\exp(z)*2*z, \exp(w)*2*w, \) etc. in a similar fashion.
Variables assigned expressions work with \( int \), but not variables assigned function definitions

\[
\begin{align*}
\text{proc}(x) & \\
& \text{if } 0 \leq x \text{ then return } x \text{ else return } -x \text{ end if} \quad \text{(18.36)} \\
\text{end proc}
\end{align*}
\]

\[
\text{int}(\text{procAbs}(b), b)
\]

Error, (in procAbs) cannot determine if this expression is true or false: \( 0 \leq b \)

\[
\begin{align*}
\text{procNew} & := \text{proc}(x) \\
\text{eval}(\exp(x) \cdot a, a = 2 \cdot x) & \\
\text{end}
\end{align*}
\]

\[
\begin{align*}
\text{proc}(x) \text{ eval}(\exp(x) \cdot a, a = 2 \cdot x) \text{ end proc} & \quad \text{(18.37)} \\
\text{int}(\text{procNew}(y+1), y) & \\
2 e^{y+1} y & \quad \text{(18.38)}
\end{align*}
\]

### 18.4 Definite integration

**Definite integration with int**

The clickable interface can be used to input a definite integration problem for Maple to compute. Use the \( \int_a^b \) item in the Expression Palette, then edit the expressions \( f, a \), and \( b \), and the variable of integration \( x \) to suit your problem.

The general form of the textual version of the command is

\[
\text{int( expression }, \text{ var } = a..b \text{ )}
\]

If the answer to a definite integration problem returns as the symbolic answer, then doing \( \text{evalf( ...)} \) of it will produce an approximation to the area, using techniques that do not require symbolic integration to find the answer. Approximation techniques will be used automatically, without \( \text{evalf} \); if the integrand or the limits of integration involve floating point numbers rather than exact fractions.

**Definition**

Integration Examples

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Anton, Calculus 8th ed. problem 6..5.14.</strong></td>
<td></td>
</tr>
<tr>
<td>Calculate ( \int_{-10}^{5} 6 , dx ), ( \int_{-\pi}^{\pi} \frac{3}{3} \sin(x) , dx ), ( \int_{0}^{3} \sqrt{4 - x^2} , dx ), ( \int_{0}^{\pi} \frac{3}{3} \sin(x) , dx ).</td>
<td></td>
</tr>
</tbody>
</table>
| \( \int_{-10}^{5} 6 \, dx \) & 90 \quad \text{(18.39)}
| \( \int_{-\pi}^{\pi} \frac{3}{3} \sin(x) \, dx \) & Using the clickable interface.
| \( \int_{0}^{\pi} \frac{3}{3} \sin(x) \, dx \) & Using the document interface but with the textual form of the operation.

**Commentary**

Using the document interface but with the textual form of the operation.
\[ \int_{0}^{3} |x - 2| \, dx \]
\[ \frac{5}{2} \quad \text{(18.41)} \]

\textsf{result := int(sqrt(4-x^2),x=0..2);}
\[ \pi \]

Using the clickable interface again.

\begin{center}
\textbf{Anton problem 32 (a). Calculate} \[ \int_{-3}^{3} \left| x^2 - 1 - \frac{15}{x^2 + 1} \right| \, dx \]. Answer should be \(30 \arctan(13/9) + 28/3\).
\end{center}

\[ \text{int(abs(x^2 - 1 - 15/(x^2 + 1)),x=-3..3)} \]
\[ \frac{28}{3} - 30 \arctan(3) + 60 \arctan(2) \quad \text{(18.42)} \]

\[ \text{simplify(1.4.4)} \]
\[ 30 \arctan \left( \frac{13}{9} \right) + \frac{28}{3} \quad \text{(18.43)} \]

\begin{center}
\textbf{Anton problem 32 (b). Calculate} \[ \sqrt{\frac{2}{1-x^2}} - \sqrt{2} \]
\end{center}

You will find it necessary to choose a good splitting point -- try plotting the function and finding where it is zero.

\[ \text{expr := abs(1/(sqrt(1-x^2)) - sqrt(2))} \]
\[ \frac{1}{\sqrt{1-x^2}} - \sqrt{2} \quad \text{(18.44)} \]

\[ \text{int(expr,x=0..sqrt(3)/2)} \]
\[ \frac{1}{2} \sqrt{\frac{2}{1-x^2}} - \sqrt{2} \quad \text{dx} \quad \text{(18.45)} \]

\[ \text{plot(expr,x=0..sqrt(3)/2)} \]

Plotting the function suggests that there's quite a change occurring around 0.7. This point is probably the split-
Zeros := solve($x=0$),x)
\[ \{ x = -\frac{1}{2} \sqrt{2}, x = \frac{1}{2} \sqrt{2} \} \] (18.46)

SplitPt := zeros[2]
\[ \{ x = \frac{1}{2} \sqrt{2} \} \] (18.47)

Bpt := eval(x, (1.4.9))
\[ \frac{1}{2} \sqrt{2} \] (18.48)

\[ \int_0^{\text{bpt}} \text{expr \, dx} + \int_{\text{bpt}}^{\sqrt{3}/2} \text{expr \, dx} \]
\[ -\frac{1}{6} \pi + 2 - \frac{1}{2} \sqrt{2} \sqrt{3} \] (18.49)

We use the splitting idea to ask Maple to evaluate two definite integrals. It can do each piece.

We predict that in 20 years Maple or similar systems may be able to do this problem completely on its own without any manual determination of splitting points.

Anton Example 4, ch. 8 Evaluate \[ \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx \]
\[ \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx = \pi \] (18.50)

Maple can do improper definite integrals since it can do symbolic limits (e.g. the functionality available through limit).

You can input infinity by through the Common Symbols Palette that can be found on the left hand side of the Maple application window.

Anton Example 6 (e) Ch 8, Evaluate

Infinity is can also be input textually as the symbol infinity.
\[
\int_{0}^{\infty} \frac{1}{\sqrt{x} \cdot (x + 1)} \, dx
\]

answer := \text{int}(1/(sqrt(x)*(x+1)), x = 0 .. infinity);
printf("sin evaluated at the answer is: %a",
    sin(answer));

\[
\pi
\]

sin evaluated at the answer is: 0

If the coefficients or limits of integration are floating point numbers, then Maple may use alternative techniques to approximate the integral. These approximations are often good and sometimes work when the exact techniques do not.

**Approximated definite**

### Approximate definite integration examples

<table>
<thead>
<tr>
<th>Expression</th>
<th>Result</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{result := int} \left( \frac{\exp(x^2)}{\log(x^2)} \cdot x = 2.3 \right) )</td>
<td></td>
<td>Maple is unable to calculate the exact solution to this definite integration problem.</td>
</tr>
<tr>
<td>( \int_{2}^{3} \frac{e^2}{\ln(x^2)} , dx )</td>
<td></td>
<td>evalf says that the area is approximately 694.15.</td>
</tr>
<tr>
<td>evalf(result)</td>
<td>694.1492846</td>
<td>Note that the difference between this and the original input was that we have changed the upper limit of integration from &quot;3&quot; to &quot;3.0&quot;. Because this is a floating point number, Maple will use evalf automatically.</td>
</tr>
<tr>
<td>( \text{int} \left( \frac{\exp(x^2)}{\log(x^2)} \cdot x = 2.3.0 \right) )</td>
<td></td>
<td>This is the same integrand. Note that the lower limit of integration is where the integrand is infinite.</td>
</tr>
<tr>
<td>( \int_{1}^{3} \frac{e^2}{\ln(x^2)} , dx )</td>
<td></td>
<td>Not surprisingly, the approximation method doesn't give us a number for the answer. Since the approximation method doesn't guarantee that its answer is close to the actual answer, this result doesn't prove that the area under the curve is infinite. Further analysis by a mathematician might establish this conclusively.</td>
</tr>
<tr>
<td>evalf((1.4.16))</td>
<td>\text{Float}( \infty )</td>
<td></td>
</tr>
<tr>
<td>( \text{int} \left( \frac{x^2 + 2x + 1}{x^2 + 2x^2 + 1} \cdot x = 0..1 \right) )</td>
<td></td>
<td>There is a symbolic solution to the integral, but it's messy. An explanation of &quot;RootOf&quot; can be found in Example 17.1.3.2.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>As with the integrand above, making the upper limit of integration into a floating point number invokes the approximation technique that evalf uses.</td>
</tr>
</tbody>
</table>
Approximate definite integration examples

\[
\begin{align*}
- & \left( \frac{\sum_{R=\text{Root}(f)} \left( R^3 + 2R + 1 \right) } { 2 \left( 5 - R + 10R^2 \right) \ln(-R) } \right) - \frac{89}{12} + \\
& \left( \frac{\sum_{R=\text{Root}(f)} \left( R^3 + 2R + 1 \right) } { 2 \left( 5 - R + 10R^2 \right) \ln(1-R) } \right) \\
& \text{im} \left( \frac{(x^3 + 2x + 1)}{(x^2 + 2x^2 + 1)}, x=0..1.0 \right) \\
& 1.161103706
\end{align*}
\]

18.5 Arc-length integration

Recall the following information about the arc length of parametric curves (Anton, ch 7.4, theorem 7.4.3). If no segment of the curve represented by the parametric equations \( x = x(t), y = y(t), a \leq t \leq b \)

is traced more than once as \( t \) increases from \( a \) to \( b \), and if \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) are continuous functions for \( a \leq t \leq b \),

then the arc length \( L \) of the curve is given by

\[
L = \int_{a}^{b} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt .
\]

We can use this formula and Maple's definite integration operation to calculate arc length easily.

Maple has an arclength operation built into its "Student1" package:

**Calculating arclength**

\[
\text{Student[Calculus1][Arclength][x \text{ expression, y expression}, \text{ parameter} = a..b]}
\]

calculates the arclength of a curve described by the parametric expressions in the two coordinates. The package uses the definite integration feature of Maple. Therefore, it will use approximate methods if \( a, b, \) or any number in the expressions is a floating point number.

\[
\text{Student[Calculus1][Arclength][f(x), x = a..b];}
\]

calculates the arc of the curve given by the parameterization \( x=t, y=f(t) \)

**Solving a problem involving arclength**

Problem 11, section 7.4 Anton p. 489: Let the parameterization be \( x = \cos(2t), y = \sin(2t), 0 \leq t \leq \frac{\pi}{2} \). Find the arclength.

\[
\text{plot} \left( \cos(2t), \sin(2t), t=0..\frac{\pi}{2}, \text{ scaling = constrained} \right)
\]
Solving a problem involving arclength

\[ \text{Student[Calculus1][ArcLength]}([\cos(2 \cdot t), \sin(2 \cdot t)], t = 0 \ldots \pi/2) \]

\[ \pi \]  

(18.58)

This is a parametric plot (see section << Brian, please provide link >> We say scaling =constrained so that the x and y axes are given the same scaling.

\[ \text{Student[Calculus1][ArcLength]}([\cos(2 \cdot t), \sin(2 \cdot t)], t = 0 \ldots \pi/2.0) \]

\[ 3.141592654 \]  

(18.59)

The use of "2.0" makes Maple use approximation techniques in the arclength calculation.

\[ npExpr := \sqrt{1 - x^2} \]

\[ \sqrt{1 - x^2} \]  

(18.60)

\( \text{plot(npExpr, x = -1..1, scaling = constrained)} \)
Solving a problem involving arclength

This is an alternative formulation for the same curve, as the plot indicates.

\[
\text{with(Student[Calculus1])}
\]

\[
\text{[AntiderivativePlot, AntiderivativeTutor, ApproximateInt, ApproximateIntTutor, ArcLength, ArcLengthTutor, Asymptotes,}
\]

\[
\text{Clear, CriticalPoints, CurveAnalysisTutor, DerivativePlot, DerivativeTutor, DiffTutor, ExtremePoints, FunctionAverage,}
\]

\[
\text{FunctionAverageTutor, FunctionChart, FunctionPlot, GetMessage, GetNumProblems, GetProblem, Hint,}
\]

\[
\text{InflectionPoints, IntTutor, Integrand, InversePlot, InverseTutor, LimitTutor, MeanValueTheorem,}
\]

\[
\text{MeanValueTheoremTutor, NewtonQuotient, NewtonsMethod, NewtonsMethodTutor, PointInterpolation, RiemannSum,}
\]

\[
\text{RollesTheorem, Roots, Rule, Show, ShowIncomplete, ShowSolution, ShowSteps, Summand, SurfaceOfRevolution,}
\]

\[
\text{SurfaceOfRevolutionTutor, Tangent, TangentSecantTutor, TangentTutor, TaylorApproximation,}
\]

\[
\text{TaylorApproximationTutor, Understand, Undo, VolumeOfRevolution, VolumeOfRevolutionTutor, WhatProblem]}
\]

\[
\text{ArcLength(npExpr, x = -1..1)}
\]

\[
\pi
\]

(18.62)

ArcLength is a function in the Student[Calculus1] package. We load in the package using \textit{with}. ...
18.6 Piecewise expressions and piecewise functions

Mathematical situations sometimes need functions that have more than one part to their definition. Consider the following expression:

\[ a := \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases} \]

This defines a piecewise expression. It means "-x if x is less than 0, x if x is \( \geq 0 \)."

\[ \begin{cases} -x & x < 0 \\ x & 0 \leq x \end{cases} \quad (18.63) \]

**Piecewise expressions**

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ a := \begin{cases} -x &amp; x &lt; 0 \ x &amp; x \geq 0 \end{cases} ]</td>
<td>Define a piecewise expression. It means &quot;-x if x is less than 0, x if x is ( \geq 0 ).&quot;</td>
</tr>
<tr>
<td>( \text{plot}(a, x=-5..5) )</td>
<td>Plot the piecewise expression for x in the range -5..5.</td>
</tr>
<tr>
<td>[ b := \text{piecewise}(x &lt; 0, -x, x \geq 0, x) ]</td>
<td>b is assigned the same piecewise expression, using the textual way of entering the information.</td>
</tr>
<tr>
<td>[ bMinus1 := \text{simplify}(2 \cdot b - 1) ]</td>
<td>Maple can do limited algebra on piecewise expressions.</td>
</tr>
</tbody>
</table>

\[ \begin{cases} -1 - 2x & x < 0 \\ -1 + 2x & 0 \leq x \end{cases} \quad (18.66) \]
We can provide more branches to a piecewise expression. In the Palette version, you can add an extra row by typing control-shift-R (on Macintosh, command-shift-R) in the bottom row. In the textual version, you can add extra items onto the sequence of inputs to the function.

### More extensive piecewise expressions with the clickable and textual interfaces

<table>
<thead>
<tr>
<th>Clickable interface</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c := \begin{cases} 1 &amp; x &lt; -1 \ 2 &amp; 1 &gt; x \geq -1 \end{cases} \rightarrow )</td>
<td>To get this we typed ( c := ), then selected the ( x &lt; a ) item from the palette. We edited what was there. Then we typed control-shift-R and filled in the bottom row. After completing the piecewise expression, we right-clicked (control-clicked in Mac) the expression and selected the 2D plot item from the popup menu.</td>
</tr>
</tbody>
</table>

![Graph of piecewise function](image)

<table>
<thead>
<tr>
<th>Textual interface</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{step2} := \text{piecewise}(x &lt; -1, 1, -1 \leq x &lt; 2, x^2, x \geq 2, x^3) )</td>
<td>This is an expression with three pieces that fit together at -1, and 2. To get the ( \leq ), we typed the characters ( \leq ) consecutively. Maple automatically reset them as the mathematical symbol ( \leq ). This uses the plot facility using the textual interface. We can evaluate the expression at ( x = 1.5 ). Since the input is limited-precision and the expression can be completely evaluated using arithmetic, we get a limited-precision result.</td>
</tr>
</tbody>
</table>

\[ \begin{align*}
1 & \quad x < -1 \\
\text{} & \text{} \\
\text{x}^2 & \quad -1 \leq x \text{ and } x < 2 \\
\text{} & \text{} \\
\text{} & \text{x}^3 \quad 2 \leq x
\end{align*} \]
It is possible to use *otherwise* as one of the clauses of a piecewise expression. Usually there are an even number of inputs to piecewise, but if there is an odd number, then the last input is the "otherwise" value.

**More extensive piecewise expressions with the clickable and textual interfaces**

<table>
<thead>
<tr>
<th>Clickable interface</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>eval[step2, x = 1.5]</td>
<td>2.25</td>
</tr>
<tr>
<td></td>
<td>(18.69)</td>
</tr>
</tbody>
</table>

To get this we typed `c := {1, -2 < x < -1, 2, 1 > x ≥ -1}` then selected the \( x \), \( x < a \) and \( x \), \( x \geq a \) item from the palette. We edited what was there. Then we typed control-shift-R and filled in the bottom row. After completing the piecewise expression, we right-clicked (control-clicked in Mac) the expression and selected the 2D plot item from the popup menu.
**Clickables**

**Textual interface**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>step3 := piecewise(-2\pi \leq x \leq \pi, \sin(x), \pi &lt; x &lt; 5, 0, -2)</code></td>
<td>We define an expression that is ( \sin(x) ) for ( x ) between (-2\pi) and ( \pi), 0 between ( \pi ) and 5, and -2 everywhere else. If the number of terms in the sequence of inputs to <code>piecewise</code> is odd, then the last term is the &quot;otherwise&quot; clause.</td>
</tr>
</tbody>
</table>

Plot:

![Plot of step3 function](image)

Since a piecewise expression is still an expression, we can use it as part of a function definition. Within a function definition, a piecewise expression can be used instead of an `if then ... end if`.

**More piecewise function definitions and uses**

<table>
<thead>
<tr>
<th>Clickable interface</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>h := x -&gt; piecewise(x &lt; 1.5, x, 1.5 \leq x, x \log(x))</code></td>
<td>To get this we selected the <code>f:=a \rightarrow y</code>, from the Expression Palette, and then filled in <code>h</code> for <code>f</code>, and <code>x</code> for <code>a</code>. Positioning the cursor at <code>y</code>, we then selected the <code>\begin{cases} \end{cases}</code> item from the <code>solve</code> operation tries to find (numerically, using limited-precision arithmetic) a value of <code>x</code> such that <code>h(x)</code> is approximately equal to 3.232 is greater than 1.5, and <code>2.954165523*\log(2.954165523)</code> is approximately 3.2.</td>
</tr>
</tbody>
</table>

`h(1.5)`

| 0.6081976622 |

`solve(h(x) = 3.2)`

| 2.954165523 |

Plot:

![Plot of h function](image)
18.7 Calculations with piecewise expressions

Many scientific and engineering situations can be modeled with piecewise expressions. Maple can apply `solve`, `diff`, and `int` on piecewise expressions, This can often save you time over trying to perform the operations on the pieces separately.

**Calculus on piecewise expressions and functions**

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
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</thead>
<tbody>
<tr>
<td><strong>Problem:</strong> An ion-powered space probe is moving through a gaseous nebula. For the first ten hours the ion engine is on, and, it travels at a rate of ( 0.05 r^2 ) meters per second. Then the motor shuts off and the gas in the nebula gradually slows the space probe down. The velocity of the space probe can be described as:</td>
<td></td>
</tr>
</tbody>
</table>
| \[
\begin{align*}
  0 & \quad t < 0 \\
  \frac{5}{100} r^2 & \quad 0 \leq t < 10 \\
  \frac{5}{100} \cdot 10^2 \exp(-t + 10) & \quad t \geq 10
\end{align*}
\] |
<p>| What is the position of the space probe at ( t=20 ) hours? |</p>
<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution</td>
<td></td>
</tr>
</tbody>
</table>
| \[
velocity := \begin{cases}
0 & t < 0 \\
\frac{5}{100} t^2 & 0 \leq t < 10 \\
\frac{5}{100} \cdot 10^2 \exp(-t + 10) & t \geq 10
\end{cases}
\] (18.77)                                                                 | We assign the piecewise expression to the variable \textit{velocity} and plot it to better understand it. The plot reveals that there are not any discontinuities in velocity, although there is an abrupt transition at \(t=10\). |
| \(\text{plot}(velocity, t=0..20)\)                                    |                                                                            |
| \[\text{position} := \int(velocity, t)\]                               |                                                                            |
| \[
\begin{cases}
0 & t \leq 0 \\
\frac{1}{60} t^3 & t \leq 10 \\
-5e^{-t+10} + \frac{65}{3} & 10 < t
\end{cases}
\] (18.78)                                                              | Since position is the integral of velocity, we can get an expression for the position by symbolically integrating velocity with respect to time. To find the position at \(t=20\), we evaluate the expression \textit{position} at that value. By giving the floating point number 20.0 instead of the exact number 20, we ensure that the result is given as a floating point number. |
| \(\text{eval(position, t = 20.0)}\)                                     | 21.66643967 (18.79)                                                       |
| \(\text{distance} := \int(velocity, t = 0..20.0)\)                    | We should get the same result if we integrate velocity for \(t=0\) to \(t=20\) |
| \(21.66643967\)                                                        |                                                                            |
Piecewise integration

Problem 4(c) Anton, problem 6.7.42. A sprinter in a 100 m race explodes out of the starting block with an acceleration of $4.0 \ \frac{m}{s^2}$, which she sustains for 2.0s. Her acceleration then drops to zero for the rest of the race. (a) What is her time for the race? Make a graph of her distance from the starting block versus time.

**Solution**

Create a piecewise expression modeling the acceleration.

$$aexpr := \text{piecewise}(0 \leq t \text{ and } t \leq 2, 4, 0)$$

$$aexpr := \begin{cases} 4 & 0 \leq t \text{ and } t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

We integrate the acceleration expression to get a velocity expression.

$$vexpr := \text{int}(aexpr, t)$$

$$vexpr := \begin{cases} 0 & t \leq 0 \\ 4t & t \leq 2 \\ 8 & 2 < t \end{cases}$$

We integrate the velocity expression to get a position expression.

$$dexpr := \text{int}(vexpr, t)$$

$$dexpr := \begin{cases} 0 & t \leq 0 \\ 2t^2 & t \leq 2 \\ 8t - 8 & 2 < t \end{cases}$$

We use `fsolve` to find approximately the time when the runner reaches 100 meters.

$$\text{finishTime} := \text{fsolve}(dexpr = 100, t)$$

$$\text{finishTime} := 13.50000000$$

We plot distance versus time to confirm the answer from `fsolve`.

$$\text{plot}(dexpr, t=0..\text{finishTime}, \text{labels} = ["time (in seconds)", "position (in meters)"])$$
Piecwise integration

Piecwise integration, continued

Problem

Two racers are in the race. One accelerates at 2m/sec^2 for 4 seconds, as described previously. The other accelerates at 2.1 m/sec^2 for 3.8 seconds. Who gets to the finish line first?

Solution

Create a procedure that turns the previous script into something that can handle other accelerations. The procedure should take two parameters, the initial acceleration rate aInit, and the period of time tInit that this acceleration lasts. After the initial acceleration, the racer is assumed to run at the final velocity attained for the rest of the race. We want the procedure to return the time to complete the race. It should also, as a side-effect, print out the plot but it should not return the plot structure as a result.

```
racer := proc( aInit, tInit)
local aexpr, vexpr, dexpr, finishTime, plotStructure;

#define acceleration
aexpr := piecewise(0 <= t and t <= aInit, tInit, 0);
#Calculate velocity and distance travelled.
vexpr := int(aexpr, t);
dexpr := int(vexpr, t);
#Calculate approximate time to finish.
finishTime := fsolve(dexpr = 100, t);

#Print out position vs. time.
print(plot(dexpr, t = 0 .. finishTime, labels = ["time (in seconds)", "position (in meters)"]));

#Return finishing time.
return finishTime;
end proc:
```
# Test the procedure on the original problem
r1 := racer(2.0, 4.0);

\[
\begin{array}{c}
\text{position (in meters)} \\
\text{time (in seconds)}
\end{array}
\]

13.5000000

r1 := racer(2.0, 4.0);
r2 := racer(2.1, 3.8);

# while the if statement is not really worthwhile if
# we only wanted to compare two racers, we are considering
# the possibility that we might want to do this
# frequently and so write for easy reuse.

if r1 < r2 then print("Racer 1 won!");
elif r1 > r2 then print("Racer 2 won!");
else print("They tied!");
end if;
Because piecewise expressions are similar to if then .. end if, it is tempting to think that they are interchangeable. One thing to keep in mind is that operations such as int evaluate all their arguments before they start to try to figure out their results. If one of the arguments involves a procedure, then the procedure needs to work with symbolic rather than numeric arguments.
The difference between piecewise and a procedure with if then else.

\[
\text{int}(\text{abs}(b), b)
\]

\[
\begin{align*}
-\frac{1}{2}b^2 & \quad b \leq 0 \\
\frac{1}{2}b^2 & \quad 0 < b
\end{align*}
\]  \hspace{1cm} (18.85)

\[
\text{procAbs} := \text{proc}(x) \\
\text{if } x \geq 0 \\
\text{then} \\
\text{return } x \\
\text{else} \\
\text{return } -x \\
\text{end if} \\
\text{end}
\]

\[
\text{proc}(x) \\
\text{if } 0 \leq x \text{ then return } x \text{ else return } -x \text{ end if}
\]  \hspace{1cm} (18.86)

\[
\text{int}(\text{procAbs}(b), b)
\]

Error, (in procAbs) cannot determine if this expression is true or false: 0 <= b

18.8 Curve fitting with splines

In scientific and engineering work, we often start with numerical measurements of a phenomenon, but need ways of knowing something about the mathematical behavior of the phenomenon for values other than those measured -- to make predictions, or to do mathematical modeling operations such as equation-solving, integration, or optimization that require formulae. A typical way of handling this is to construct a model (a formula) from the data.

Linear least squares data fitting, described in <<Brian, provide link to least squares section>> is a way of doing this. However, it can only find formulas that are lines. While there are techniques for non-linear least squares for certain kinds of models-- linear, quadratic, exponential, logarithmic, etc -- they all require that the math work out to make the computation feasible. Furthermore, it mandates the kind of relationship, which the scientist or engineer may not know.

There is a different kind of formula-fitting called spline interpolation. A linear spline is a piecewise expression where each piece is a line. The spline agrees with the data at each point given, and uses the line connecting two consecutive points as the formula used for any value of the controlling variable between those two points.

**Linear splines**

We enter a collection of data values

\[
dataValues := [[1, 3], [1.5, 2.8], [2.1, 4.6], [3.0, 4.4]]
\]  \hspace{1cm} (18.87)

... and plot them.

\[
dataPlot := \text{plot}(dataValues, style = \text{point}, \text{view} = [1.4, 0.5], \text{symbol = diamond}, \text{symbolsize = 20, color = "Purple"})
\]  \hspace{1cm} (18.88)

\[
\text{print}(dataPlot)
\]
We load in the CurveFitting package, and construct a linear spline which we call \( s \). Without the with, we could just do CurveFitting[Spline].

\[
\text{with(CurveFitting)}
\]

\[
\begin{array}{ll}
\text{[ArrayInterpolation, BSpline, BSplineCurve, Interactive, LeastSquares, PolynomialInterpolation, RationalInterpolation,}
\end{array}
\]

\[
\text{Spline, ThieleInterpolation]}
\]

\[
s \equiv \text{Spline(dataValues, t, degree = 1)}
\]

\[
\begin{align*}
3.400000000 - 0.4000000000 t & \quad t < 1.5 \\
-1.700000000 + 3.000000000 t & \quad t < 2.1 \\
5.066666667 - 0.2222222222 t & \quad \text{otherwise}
\end{align*}
\]

We display a plot of the spline, with the data point plot. This shows that the spline \( s \) does agree with all the data points.

\[
\text{plots[display][[plot(s, t = 0..4), dataPlot]]}
\]
To get a prediction of what to expect for say \( t = 2.3 \), we just evaluate \( s \) at that point.

\[
eval(s, t = 2.3) = 4.55555556
\]  

(18.91)

While we don't know how accurate this will be, we know that if the relationship being modeled is reasonably smooth, then this shouldn't be far off.

We can apply the standard math operations to the piecewise expression that the spline generator produces.

\[
\text{Optimization[ Maximize ]}(s, t = 0..3.1)
\]

\[
\left[ 4.5999999843951990, [t = 2.10000000873216131] \right]
\]  

(18.92)

\[
\text{int}(s, t = 0..3.1)
\]

\[
8.15888889
\]  

(18.93)

\textit{fsolve} will find one solution, but if we want the solution to be inside the range of values that we measured, we should specify the interval

\[
\text{fsolve}(s = 3.1)
\]

\[
0.7500000000
\]  

(18.94)

\[
\text{fsolve}(s = 3.1, t = 1..4)
\]

\[
1.6000000000
\]  

(18.95)

One problem with linear splines is that they can have sharp corners, as demonstrated by the example just given. It is possible, with sufficient data points, to construct splines where the pieces are quadratic, cubic, or of even higher degree. This has the advantage that the bends in the spline will be smoother. How smooth depends on the degree. However, higher degree splines may do undesirable things for values outside the ranges being measured. For that reason, it's better to use splines for problems where you need results for values \textit{inside} the ranges being measured.
Quadratic splines and splines of higher degrees

The alternative form of spline construct takes two lists, one for first coordinate, and one for the second.

\[ tValues := [1, 1.7, 2.1, 3.7, 4.1] \]
\[ mValues := [3, 2.7, 4.8, 4.4, 3.1] \]

We can plot them.

\[ dataPlot2 := plot(tValues, mValues, style = point, view = [1..8, 0..8], symbol = diamond, symbolsize = 20, color = "Purple") \]

\[ \text{PLOT(...)} \]

\[ \text{print(dataPlot2)} \]

With load in the CurveFitting package, and construct a linear spline which we call \( s \). Without the with, we could just do CurveFitting[Spline].

\[ \text{with(CurveFitting)} \]
\[ [\text{ArrayInterpolation, BSpline, BSplineCurve, Interactive, LeastSquares, PolynomialInterpolation, RationalInterpolation, Spline, ThieleInterpolation}] \]

\[ s2 := \text{Spline(tValues, mValues, t, degree = 2)} \]
\[
\begin{align*}
3. & - 3.5097532970000005 \, (t - 1)^2 & t & < 1.3500000000000009 \\
-2.738927138 & + 3.1993689050000004 & t & + 8.08028030500000050 \, (t - 1.7)^2 & t & < 1.8999999999999992 \\
-5.569104990 & + 4.93766904199999956 & t & - 3.7345299569999990 \, (-2.1 + t)^2 & t & < 2.8999999999999992 \\
18.69129170 & - 3.86251127000000016 & t & - 1.76558273900000007 \, (t - 3.7)^2 & t & < 3.8999999999999992 \\
3.100000000 & + 11.42186099999995 & (t - 4.1)^2 & otherwise
\end{align*}
\]

We display a plot of the spline, with the data point plot. This shows that the spline \( s \) does agree with all the data points.

```maple
plots[display]([plot(s2, t = 0 .5), dataPlot2])
```

To get a prediction of what to expect for say \( t = 2.3 \), we just evaluate \( s \) at that point.

```
eval(s, t = 2.3)
```

```
4.55555556
```

The formula does have the drawback that it starts increasing after \( t = 4.1 \), even though the data doesn't. This is a property of higher degree polynomials. We might not want to use this model if we need to extrapolate very far beyond the values that were measured. Higher degree spline interpolation does not necessarily do a lot better in extrapolation, although they all agree fairly closely for values within the \( t \) values measured.

```
s3 := Spline(tValues, mValues, t, degree = 3) :
s4 := Spline(tValues, mValues, t, degree = 4) :
plots[display]([plot(s2, t = 0 .5, color = red), plot(s3, t = 0 .5, color = green), plot(s4, t = 0 .5, color = blue), dataPlot2])
```
18.9 Chapter summary

**Indefinite integration with int**

The clickable interface can be used to input an indefinite integration problem for Maple to compute. Use the $\int f \, dx$ item in the Expression Palette, then edit the expression $f$ and the variable of integration $x$ to suit your problem.

The general form of the textual version of the command is

```
int( expression, var )
```

This computes the anti-derivative of the expression, with respect to the variable $var$. If the answer is just a formula involving expressions that the user already knows about (e.g. ones that involve numbers, polynomials, rational functions, trig, logarithm and exponential functions), when Maple can find one, is sometimes referred to as a _closed form solution to the integration problem._

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int \frac{1}{x} , dx$</td>
<td>The clickable interface can be used to enter the integration problem. Note that in the clickable interface there are two ways of entering what is &quot;$\cos(x) \cdot \cos(x)$&quot;.</td>
</tr>
<tr>
<td>$\int \cos^2(x) , dx$</td>
<td>$\frac{1}{2} \cos(x) \sin(x) + \frac{1}{2} x$ (18.103)</td>
</tr>
<tr>
<td>$\int \cos^2(x) , dx$</td>
<td>$\frac{1}{2} \cos(x) \sin(x) + \frac{1}{2} x$ (18.104)</td>
</tr>
</tbody>
</table>
\[
\text{\texttt{int(sqrt(x),x)}} \\
\frac{2}{3} x^{3/2} \quad (18.105)
\]
\[
\text{\texttt{int(\cos(\alpha) \cdot \sin\left(\frac{\alpha}{2}\right), \alpha)}} \\
-\frac{1}{3} \cos\left(\frac{3}{2} \alpha\right) + \cos\left(\frac{1}{2} \alpha\right) \quad (18.106)
\]

```
expr := tan(t) + x^2;
il := int(expr,t);
dl := diff(il,t);
difference := dl-expr;
print("difference":. difference);
print("simplified difference":,
simplify(difference));
```

\[
\tan(t) + x^2 \\
-\ln(\cos(t)) + x^2 t \\
\frac{\sin(t)}{\cos(t)} + x^2 \\
\frac{\sin(t)}{\cos(t)} - \tan(t) \\
"difference":\left(\frac{\sin(t)}{\cos(t)} - \tan(t)\right) \\
"simplified difference":, 0
\]

### Definition Integration examples

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculate ( \int_{-10}^{5} 6 , dx ), ( \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin(x) , dx ), ( \int_{0}^{3}</td>
<td>x - 2</td>
</tr>
<tr>
<td>( \int_{-10}^{5} 6 , dx )</td>
<td>( 90 ) \quad (18.107)</td>
</tr>
<tr>
<td>( \text{\texttt{int(\sin(x),x=\frac{\pi}{3},\frac{\pi}{3})}} )</td>
<td>( 0 ) \quad (18.108)</td>
</tr>
<tr>
<td>( \int_{0}^{3}</td>
<td>x - 2</td>
</tr>
</tbody>
</table>
\[
\int \sqrt{4-x^2} \, dx, x=0..2;
\]

\[\pi\]

**Anton problem 6.6.25.** Calculate \[
\int_{1}^{4} \left( \frac{3}{\sqrt{t}} - 5\sqrt{t} - t^{\frac{3}{2}} \right) dt
\]

\[\text{int}(3/\sqrt{t}) - 5*\sqrt{t} - t^{\frac{3}{2}}, t = 1..4)\]

\[-\frac{55}{3}\]  

(18.110)

**Anton problem 32 (a).** Calculate \[
\int_{-3}^{3} \left| x^2 - 1 - \frac{15}{x^2 + 1} \right| dx
\]

Answer should be \(30 \arctan(13/9) + 28/3\).

\[\text{int(abs(x^2 -1 - 15/(x^2 + 1)), x=-3..3)}\]

\[\frac{28}{3} - 30 \arctan(3) + 60 \arctan(2)\]  

(18.111)

\[
simplify((1.10))
\]

\[30 \arctan \left( \frac{13}{9} \right) + \frac{28}{3}\]  

(18.112)

**Anton Example 4, ch. 8** Evaluate \[
\int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx
\]

\[\frac{\pi}{2}\]  

(18.113)

**Anton Example 6 (c) Ch 8.** Evaluate \[
\int_{0}^{\infty} \frac{1}{\sqrt{x} \cdot (x + 1)} \, dx
\]

\[
\text{answer := int(1/(sqrt(x) * (x+1)), x = 0 .. infinity);} \\
\text{printf("sin evaluated at the answer is: \%a", sin(answer));}
\]

\[\pi\]

sin evaluated at the answer is: 0

Approximate definite integration examples
\[
\int \frac{\exp(x^2)}{\log(x^2)} \, dx, \quad x = 2.3
\]
\[
\int_2^3 \frac{e^{x^2}}{\ln(x^2)} \, dx \quad (18.114)
\]
\[
evalf((1.9.13))
\]
\[
694.1492846 \quad (18.115)
\]
\[
\int \frac{\exp(x^2)}{\log(x^2)} \, dx, \quad x = 2.30
\]
\[
694.1492846 \quad (18.116)
\]
\[
\int \frac{\exp(x^2)}{\log(x^2)} \, dx, \quad x = 1.3
\]
\[
\int_1^3 \frac{e^{x^2}}{\ln(x^2)} \, dx \quad (18.117)
\]
\[
evalf((1.9.16))
\]
\[
\text{Float} (\infty) \quad (18.118)
\]
\[
\int \left( \frac{x^2 + 2x + 1}{x^2 + 2x^2 + 1} \right), \quad x = 0.1
\]
\[
\sum_{R=\text{RootOf}(\sqrt{3} + 2 - \sqrt{2} + 1)} \frac{2(5 - R + 10 R^2) \ln(-R)}{R(3R + 4)} - \frac{89}{12} + \quad (18.119)
\]
\[
\int \left( \frac{x^2 + 2x + 1}{x^2 + 2x^2 + 1} \right), \quad x = 0.10
\]
\[
1.161103706 \quad (18.120)
\]

### Calculating arc length

\[ \text{Student}[\text{Calculus1}][\text{Arclength}][\{x \text{ expression}, y \text{ expression}\}, \text{parameter} = a..b] \]

calculates the arc length of a curve described by the parametric expressions in the two coordinates. The package uses the definite integration feature of Maple. Therefore, it will use approximate methods if \( a, b \), or any number in the expressions is a floating point number.

\[ \text{Student}[\text{Calculus1}][\text{Arclength}][f(x), x = a..b]; \]
calculates the arc of the curve given by the parameterization \( x=t, y=f(t) \)

\[ \text{Student}[\text{Calculus1}][\text{Arclength}][f(x), x = a..b, \text{showfunction}=\text{true}]; \]
calculates the arc of the curve given by the parameterization \( x=t, y=f(t) \)
A piecewise expression uses different expressions to describe the value for different ranges of the controlling variable. It can be entered either with the clickable interface or textually.

To add more branches to a piecewise expression. In the clickable version, you can add an extra row by typing control-shift-R (on Macintosh, command-shift-R) in the bottom row. In the textual version, you can add extra items onto the sequence of inputs to the function.

If the word otherwise is used for a range in the clickable interface, then the expression associated with that branch will be used for all other values of the controlling variable. The otherwise branch is specified in the textual version of piecewise as the odd last argument to piecewise expression.

A difference between piecewise expressions and if then ... end if is that Maple can do mathematical operations such as +, solve, diff, int, or plot on piecewise expressions, while it can't do so as easily with if then ... end if.

\[
h(x) := \begin{cases} 
  x & x < 1.5 \\
  x \cdot \log(x) & x \geq 1.5
\end{cases} \quad x \rightarrow \text{piecewise}(x < 1.5, x, 1.5 \leq x, x \cdot \log(x))
\]

\[
h(1.5) = 0.6081976622 \quad (18.122)
\]

\[
solve(h(x) = 3.2)
\]

\[
2.954165523 \quad (18.123)
\]

\[
\text{step3 := piecewise}\{-2 \leq x \leq \pi, \sin(x), \pi < x < 5, 0, -2\}
\]

\[
\begin{array}{c}
\sin(x) & -2 \leq x \text{ and } x \leq \pi \\
0 & \pi < x \text{ and } x < 5 \\
-2 & \text{otherwise}
\end{array} \quad (18.124)
\]

\[
\text{plot(step3, x=-10 .. 10)}
\]
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