Scripting and Programming for Modeling, Simulation, and Control

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Scripting and Programming for Modeling, Simulation, and Control
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Dedication

To our students, who learn how to work with the new and different.

To my family, whose support is unwavering.

Acknowledgements

Jeremy Johnson started the Computation Lab course. He chose the initial direction and approach of the course. Frederick W. Chapman worked with us in 2006-2008 in the development of the course.

We are grateful to our colleagues David Augenblick, Jeremy Johnson, and Brian Dolhansky for their help in the development of this text. David was a careful proofreader and useful sounding board. Jeremy's shrewd counsel and feedback often led to improvements. Brian's expert and speedy work in tagging and repairing the manuscript kept us on schedule. We also appreciate the assistance of Maplesoft for technical advice and support, especially that of Christina Spirou.
1 Introduction -- Technical computing at the turn of the century

1.1 Chapter synopsis

1. We explain what technical computing is about, why it's done with a machine and what kinds of devices are used to do it.

2. The advantages doing calculation with a computer are surveyed.

3. Maple, a system for technical computing is introduced and contrasted with some of the other choices available today.

1.2 What is technical computing? What kinds of technical computing are there?

Well, obviously, it's "computing done for technical work" -- primarily science and engineering, but any other field where mathematical and scientific reasoning is used, which nowadays can include other fields such as finance and business, medicine, or digital media. The success of such reasoning in these fields is well-known. Students of those fields need to become experienced and develop proficiency in how it's done to be able to obtain success in the same way.

You are probably familiar with numerical computing since it's done even without computers, with paper and pencil or with calculators in high school or even earlier. You probably have also seen graphing calculators do simple kinds of graphical computing, also sometimes known as mathematical visualization. In addition to this, computers can and are used to do symbolic computing -- computing where the answer is a formula rather than a number, as well as deductive or logical computing -- using logic or deduction to find answers.

In this course, we will concentrate on numerical, symbolic and graphical computing, although we will see some instances of deductive computing as well.

1.3 What are the advantages of doing technical calculation with a computer?

One advantage is greater quantity of calculation-- computers can do calculations billions of times faster than humans and thousands or millions of times faster than calculator-driven computation. Using them makes some things feasible that are not possible any other way:

As you've seen with your high school physics, chemistry, and math courses, scientists and engineers often work with mathematical models -- systems of equations, symbols, and mathematical relations that try to describe key aspects of a situation. Calculation with models helps answer questions about the situation being modeled: How much fuel will be used in this operating scenario? How long will it take to heat up the furnace to 265 degrees? Computer simulation loads a model with some initial conditions, and then through calculation produces predictive results. Simulation can be used to make forecasts such as: how fast will the vehicle be moving after 5 seconds? How far will the pollutants move underground after being buried five years? Computer simulations can sometimes generate predictions even when standard techniques of "mathematical solution" are not adequate to find an answer.

Computers make it possible to use models that require much more extensive calculations to reach conclusions. The payoff may be better predictions. Extensive calculation also makes it possible to generate and present more information. Computer visualization goes beyond that to highly detailed pictures or animations of a situation described through a mathematical model. This can lead to better insight and understanding.

Another advantage is that it is easier to make a lasting written record of the work. One of the big differences between professional technical work and homework is that the modeling and calculation has lasting value: it matters to more than one or two people, and it is being used in an on-going project. In a professional situation, a worker may need to go back and review the work a year later, long after the details are faded from easy recall. Other people may want to reuse the programming and will need to be given an explanation about how to use it. Since they have long-term value, it is beneficial to put them into a form suitable for easy future reference and reuse. Thus the work typically includes both programming and documentary explanation.
1.4 The spectrum of devices for technical computing

Back in the days of the original electronic computers developed during and immediately after World War II, the only kind of device you could use for technical computing (which was the only computing that could be justified in those days) filled up a whole room and needed an extensive staff to support.

Table 1.1: An early computer

![ENIAC, one of the first electronic computers, being programmed by Herman Goldstine at the University of Pennsylvania circa 1946. (US Army photo) Originally intended to do artillery calculations, it was used for the design and development of the first thermonuclear weapons. See http://www.seas.upenn.edu/~museum/.

Today's users have a choice of a wide variety of devices:

**Personal computers**

Typically a computer for individual use can be expected to have the following features:

1. Processing capability of a billion or more arithmetic or memory operations per second. Significant amounts of calculation are used in doing the graphics involved in processing digital media or supporting the artificial worlds presented in computer games. This same power can be used to do the simulation and exploration in many common kinds of scientific and engineering situations. This course is about using that calculation power.

2. Standard selection and pointing devices, keyboard and mice. Some computers support "tablet" operation which allow use of a pen writing on the display. The display area and input devices make it easy to enter information via text or by pointing/selection. While the World Wide Web has made much processing possible with just "pointing and clicking", the complexity and mathematical sophistication of the models used in current technical computing outstrip the capabilities of current "point and click" technology. Some (computer programming) language-based interaction is necessary to handle things deftly. This requires a more knowledgeable user, but many people find that the extra power is worth the learning effort.

3. A screen capable of displaying information equivalent to one or more 8 1/2 x 11 inch pieces of paper. This helps support the development of documentation, or of more complex visualizations.

4. Local storage capable of storing a significant fraction of the text of the books in the Library of Congress. Of course, it's easy for a personal media collection to fill up this kind of storage, but it should not be forgotten that the billions of characters that this storage represents can store years or decades of efforts of computer programmers.

5. Connection to the internet, which makes it easier to communicate and share work with others, and to download new programming. From the viewpoint of technical computing, that it's easy to import and run substantial amounts of programming.
High performance computers, also known as "supercomputers"

While hand held devices and personal computers are used widely by the general public for non-technical purposes, there are a class of computers that are used primarily for mathematically-based computing. Typically instead of billions of numerical operations per second ("gigaflops" -- billions of floating point operations), they employ multiple processors in parallel to do trillions of operations ("teraflops") or quadrillions ("petaflops"). The hardware, personnel and energy costs of such computers are significant, so typically they are available through centers used through the Internet from many locations. The kinds of technical problems such devices are used for would include: simulation of complicated physical situations, such as for climate prediction, astrophysics, or engineering design (aircraft, groundwater, large building, automobile).

Using such large devices incur significant expense, from hardware, software and support staff costs, as well as electrical power consumed. For example, the U.S. Department of Energy has an IBM computer at its Los Alamos National Laboratory in New Mexico, USA capable of 1 quadrillion (\(10^{15}\)) numerical operations per second. It consumes 2345.50 kilowatts when it is running. (reference: http://blog.enterpriseitplanet.com/green/blog/2008/06/green-petaflop-ibms-roadrunner-wins-supercomputer-top-spot.html) The National Center for Computational Sciences (NCCS) at Oak Ridge National Laboratory in Tennessee, USA which has several large supercomputers, has a 2008 annual budget between $80 million and $100 million. (reference: http://news.cnet.com/8301-13772_3-9985500-52.html). A typical personal computer might be capable of approximately 4 billion numerical operations per second, so the Blue Gene computer at Los Alamos provides roughly \(2.5 \times 10^6 = 250,000\) times more computing power.

Typically the programming is developed on personal computers, then moved to the larger devices. Usually the expensive supercomputer time is spent primarily on numerical computation, rather than on providing a nice-to-use interface for users. However, the results of a supercomputer may be shipped over the Internet to a personal computer so that a scientist or engineer may mull over the results in a more contemplative way without incurring additional supercomputer costs.

Multiple computers may be linked together over the Internet to get all the pieces of a particular elaborate computation done. This may have the supercomputers performing the massive numerical computations, while other smaller or personal computers linked in may be displaying the results to a distributed team of investigators who are collectively digesting results and steering the ongoing work.

Hand held or mobile devices

Calculators are useful for casual computation, where one wants to figure out the solution to a small problem once. It's easy to punch in a few numbers and operations and to read the answer on the display. As inexpensive and small mobile devices, these are typically more limited in:

1. memory (limits to computation size and to built-in features)
2. energy consumption (slower processor speed)
3. form factor (keyboard and display too small for more than casual technical use)

Typically calculators are not networked, making it harder to share results to transfer them elsewhere to continue the work.
The TI-Nspire with CAS is a recent generation calculator from Texas Instruments. It can do numerical and symbolic calculations, as well as graphing. According to ticalc.org (http://www.ticalc.org/basics/calculators/ti-nspire-cas.html) it has 16Mb memory, 20Mb storage and has a 150MHz processor. This makes it have about 100 times less memory, 8000 times less storage, and is about twenty times slower than a typical laptop with a dual core 1.5GHz processor, 2Gb memory, and 160Gb disk. Its screen is 240 x 320 pixels, giving it about twenty times less display area than a typical laptop. This is one of the first generation of calculators with limited wireless networking available as an add-on.

Smartphones, personal digital assistants (PDAs), media players have in theory the same processing capabilities as calculators although most of them are not used extensively for technical computation. One advantage of these kinds of devices is that are typically networked so that it's possible to get a more powerful computer somewhere else in the Internet "cloud" of computational resources to do some of the work.

In the future there may be more convergence of the capabilities of all "mobile small form factor" devices. However the small form factor and the limits to energy consumption will probably continue to constrain the capabilities of such devices compared to larger ones.

**Dedicated controllers**

Even some kinds of toasters have microprocessors in them nowadays. "Smart homes" may network many appliances and home features such as HVAC and lighting control. While these devices are merely configured rather than programmed by end users, students of technical computing should not forget that this is another place where programming is necessary. The economic reasons for switching to computer control of devices appear to be the greater flexibility and variety of control that can be developed at modest cost through programming for many kinds of devices. The kind of programming done for device control often has a mathematical basis. Although the processors in dedicated devices are typically a few orders of magnitude slower than those of personal computers or even smartphones, the programming languages used for them are often the same or similar to those used in personal computing.
1.5 Maple, a system for technical computing

In this course, you will learn how to do various kinds of technical tasks using Maple. The first version of Maple was developed in the early 1980s at the University of Waterloo in Ontario, Canada, but has since undergone many refinements and extension. (In Fall 2010, we will be using Maple 14.) Maple's original emphasis was on algebraic computing, so it was used by those who needed formulas as results for their work. In the early '90s it was used extensively in calculus classes as a way of supporting exploratory experimentation and more extensive experience with "applied" problems. More recently it has expanded its domain into the symbolic, numeric, and graphical calculations done more generally in scientific and engineering modeling and simulation.

Today, Maple supports numerical computing and graphical visualization about as well as it does symbolic computing. It has its own programming language but also has ways of doing calculations through the graphical user interface (GUI) that is a kind of augmented "point and click". The default way of interacting with Maple also allows you to mix documentation and computing instructions and results. Thus it's particularly easy in Maple to produce documents that produce a well-documented solution -- a description of the problem, the explanation for how to solve it, and the computations that produce the details of the answer and evidence to justify its correctness.

1.6 What about Systems X, Y, Z, ...?

There are a variety of systems and programming languages used for technical computing nowadays. In a large technical establishment such as a university or research lab, one might find, in addition to Maple the use of systems such as: Python, Matlab, C, Mathematica, Java, Octave, Macsyma, Sage, Axiom, or Fortran.

The vast array of tasks computers can be used for, the multiplicity of philosophies of system design, and the rapid and unrelenting nature of progress in computing means that there is no "best system" for any situation. Each system has its technical strengths. There may also be historical, cultural, or economic reasons why certain systems are used in certain places which co-exist with the technical justification. As with all popular and demanding activities, there are many products in use.

One important ramification of this is that students in technical fields must expect to become familiar and proficient in more than one programming system. Studying more than one brings greater knowledge of what's possible, and the advantages in judgment that comes from knowledge of diverse ways of achieving a goal. It also allows one to function effectively as a team member in larger cross-organizational efforts and to better to cope with the diversity in computing cultures that's out there.

1.7 Why pick Maple as the first system?

1. It is an interactive system, facilitating quick exploration of new ideas. Compared to languages such as C++ or Java, one can immediately start up an interactive system and calculate results through "point and click" and a little typing. It's also possible to enter computational scripts, which are sequences of steps that are less elaborate to set up than programs. The scripts can be easily re-run on variants of the original situation just by changing a line or two in the script. This allows convenient "what-if" exploration, where a number of different scenarios are explored through computation. The worksheet interface to Maple allows sophisticated mathematical typography and graphics as well as mouse- and palette- driven input.

2. It can handle calculations with formulas. Rather than figuring out the formula yourself, you can even get Maple to calculate the formula for you. While many mathematical calculations produce numerical results, formulas are needed to specify how those results are produced. Designers or developers often need to find and create the formulas rather than to just copy them from a book. A system where formulas are easy to represent and easy to create supports this kind of work. A system that allows you to represent and calculate both numbers and formulas makes it easier to do and document technical calculation than a system that just works with numbers.

3. It supports a variety of data structures that support technical computation: formulas, equations, functions, sets, lists, tables, vectors, and matrices. Rather than having calculations done with a number of digits that's fixed forever, it is easy to change the precision if more (or less) precision is needed. Having higher-level interaction with the computer is usually more productive of human time. Having all of these entities as "first class entities" in the system requires less mental effort by the user, since they
do not have to translate what they are thinking about (e.g. a table of formulas) into the terms that the computer is using. Rather, the computer language accommodates the human style of thought.

4. **It supports documentation as well as calculation.** From the instructor's point of view, it's easier to create documents that explain Maple because we can use Maple both to handle the calculation and the presentation of results. For students, having a file with both the directions that cause the computation and the results makes it easier to present the work cleanly for grading, and for future reference. For professionals doing technical work, having an **integrated environment** where text, programming and results can be combined together can be a convenience.

5. **It has a "conventional programming language".** An objective of this course is to make you become familiar enough with some of the standard elements of programming (e.g. assignments, conditionals, loops, procedure definitions) so that you can use them creatively to handle certain common situations that can't be handled through point-and-click or built-in operations. These elements will be found in highly similar form in the languages used by many other systems used for technical work.

6. **The mathematics of modeling and simulation is an explicit feature of the language.** While its programming language is conventional, Maple's language has an important added plus -- you can work over expressions, equations, functions, and other mathematical objects just as conveniently as if they were numbers or characters. Conventional languages (e.g. Java) can represent arithmetic operations -- multiplication, addition, subtraction, and division -- but after that there are special tricks and conversions that you must perform to bridge the gap between what is written in the program, and the mathematical ideas that you are trying to use in the model. It's more straightforward in Maple to represent the mathematical model and the computation based on it. This ease of expression and comprehension by programmers has a hard-headed dollars and sense payoff. Less programmer time can be spent developing a computation. There is also an efficiency advantage to explicit representation of formulas: it is possible to use systems such as Maple to automatically improve the quality and efficiency of simulation calculations. This again leads to lower costs of doing the work.

We think these things provide a software engineering advantage that will lead most technical computation systems to eventually have such functionality built-in into them.

### 1.8 Using more than one system

Any user of computers who expects to use them professionally for design and investigation must expect to eventually learn multiple systems. Using computer applications for work is like using tools in a workshop -- you would not expect to use one tool to do all tasks, even if the tool, like Maple, has "Swiss Army Knife" capabilities. Features that make it easier to do certain kinds of things may slow down doing other things. Even when systems overlap in features, they typically have differing philosophies and different technical strengths, which means that certain kinds of work may be significantly easier in one system than another. For example, developing something in Mathematica or Maple may be fine and quick for a personal computer, but making the same programming work on a supercomputer may take a lot of effort in a different language. Yet a work environment with multiple languages need not be overwhelmingly complex. Most systems with major development effort behind them (such as Maple and those mentioned in the "section above) have many similarities.

What makes things work out is this: at the introductory level, the difference between casual computing and professional technical computing is the style of working (higher emphasis on documentation, justification and ease of reuse), and the use of language-based commands/programming needed to do the more sophisticated operations in technical work. "Crossing over" to the professional mode of operation means getting over the hurdles of learning the new style of work, and learning how to interact with computers in a typical computer language. Once this hurdle is passed, it should take only incremental effort to acquire expertise in the second, third, or nth technical system. Having formal instruction on the first system should provide an explicit introduction into the concepts and the work processes to facilitate this.

Most systems realize that they cannot be the sole provider of technical computing services. If Institution A uses Mathematica and Institution B uses Python for their work, then if they expect to use each other's efforts, there has to be a way of interconnecting programs written in one system with that of another. Thus most systems have interconnections. For example, the Matlab Symbolic Toolkit allows Matlab users to call Maple to do formula manipulation. Similarly, Maple users can link to Matlab and run a Matlab program they've gotten from a colleague as part of a problem-solving process written in Maple.
Knowledge of basic programming and the concepts of *software development* make it possible to switch between systems with only a modest amount of additional effort. Software interconnection allows one to reuse programming done in another system without having to translate into another language. Symbolic computation systems like Maple also have the additional bonus of being able to translate some of their programming between languages. There is a "convert-to-Matlab" feature for example, or a "convert-to-C" feature for computations involving just numbers or text.
2 Getting started with Maple's Document Mode: doing technical work with a clickable interface

2.1 Chapter synopsis

1. How to start up Maple and perform simple calculations, algebra, and plots within it.
2. Detecting and fixing typographical mistakes.
3. Introducing mistakes caused by vocabulary misunderstandings and the use of incorrect logic in giving directions.
2. How to save Maple work so that you can refer to it or resume working on it later.
3. How to recover a Maple worksheet if it or your computer crashes.

2.2 Starting up Maple, getting a fresh start

Start up the Maple application (this varies on the type of computer system you have, typically it involves clicking or double-clicking on the Maple 13 icon, but if you can't figure it out yourself ask for a demo for someone who knows). Once the Maple application window appears, a new "document" will appear in the main working area of the Maple application. A flashing cursor will appear with the outline of a small rectangle with dashed lines. The entry mode will read "Math" and "2D Math". You can make the "quick help" black box disappear by clicking on the "close box" circle-X in the upper right hand corner.

Table 2.1: Maple started up with new document in Windows XP
After you close the quick help box, you will see the Maple cursor in a small rectangle with a dashed line outline.

Table 2.2: Maple document with first entry area

At this point, what you type will appear in the small rectangle and be regarded as a mathematical expression. In the next section, we describe what to type in order to get something useful to happen.
2.3 Evaluating an expression involving exact arithmetic

Grade school arithmetic

In the math area, type 2 + 3. "2 + 3" is regarded as a mathematical expression by Maple when you type it into the dashed rectangle. As you are typing, the input indicators should say that you are entering Math mode, in "2D Math" input, using "Times New Roman" font:

Table 2.3: Maple input using 2d math

![Maple input using 2d math](image)

This expression should show up in the work area. When you hit the enter key, then Maple will evaluate the expression. After the expression is evaluated, you should see the result displayed below the input, as in the figure below:

Table 2.4: Maple input with labeled result

![Maple input with labeled result](image)

Maple has automatically calculated the answer and given it a label (1). After Maple calculates the answer, the cursor should appear below the result in another dashed-line rectangle. This indicates that Maple is ready to do another calculation.

Maple supports all the basic arithmetic operations in a fashion similar to many other programming languages. One thing that takes some getting used to is that * (asterisk) is used to input multiplication rather than "x" or a centered dot. Another thing that takes some getting used to is that what you type is formatted to look like math notation. Thus if you type a /, Maple understands that you are talking about division and immediately starts formatting your input as if it were a fraction. If you type an asterisk, Maple formats that into a dot (⋅). There is also formatting that occurs with caret (^) since that is the way you enter an exponent in Maple.
Table 2.5: Arithmetic Operations in Maple

<table>
<thead>
<tr>
<th>Operation</th>
<th>Character to type/ character name</th>
<th>Notes</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>addition</td>
<td>+ (&quot;plus&quot;)</td>
<td></td>
<td>2 + 2</td>
</tr>
<tr>
<td>multiplication</td>
<td>* (&quot;asterisk&quot;)</td>
<td>Typing an asterisk makes a center dot(·) appear in the displayed expression.</td>
<td>2·3</td>
</tr>
<tr>
<td>division</td>
<td>/ (&quot;slash&quot;)</td>
<td>Typing a slash draws a baseline and then positions the cursor in the denominator. Subsequent typing appears in the denominator. To get out of the denominator and return to normal typing, use the right-arrow key (→). Multiple divisions are by default conducted left-to-right.</td>
<td>( \frac{2}{6} )</td>
</tr>
<tr>
<td>subtraction</td>
<td>- (&quot;dash&quot; or &quot;hyphen&quot;, typically on the same keyboard key as the underscore. Dash and underscore do not mean the same thing.)</td>
<td>Multiple subtractions are conducted leftmost first.</td>
<td>3 − 5</td>
</tr>
<tr>
<td>parentheses</td>
<td>(, ) (&quot;left parenthesis&quot;, &quot;right parenthesis&quot;)</td>
<td>Use parentheses to change the order of calculation. They are also good for removing any guesswork by the reader as the order of operations.</td>
<td>( (2 + 3) \cdot 5 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( 3 - \left( 5 - \frac{2}{6} \right) )</td>
</tr>
<tr>
<td>negation</td>
<td>- (&quot;dash&quot; or &quot;hyphen&quot;, typically on the same keyboard key as the underscore). This is the same symbol as used for subtraction</td>
<td>Put a dash in front of a number or parenthesized expression to negate it.</td>
<td>−(3·5 − 2)</td>
</tr>
<tr>
<td>power</td>
<td>^ (&quot;caret&quot;, typically on the same keyboard key as the number 6)</td>
<td>Typing a caret moves the cursor to the exponent position. Subsequent typing appears in the exponent. To get out of the exponent and return to the baseline, use the right-arrow key (→).</td>
<td>( 2^3 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( 2^3 - 5 )</td>
</tr>
</tbody>
</table>
is the product of all the integers between 1 and \( n \). It is useful in computations that compute the number of possible ways that something could happen. For example, the number of possible orderings of a deck of playing cards is \( 52! \).

Let's try out some of these operations. We can enter a sum of two fractions by using "+" and "/". If we type "2/3 + 5/6 enter", this is what we see:

\[
\frac{2}{3} + \frac{5}{6}
\]

\[
\frac{3}{2}
\]

The way to get a fraction in is to type a slash (/). As soon as you do so, Maple draws an underscore and positions the cursor underneath the fraction line. The next characters you type appear as the denominator. If you type the "+" right after the "3", the plus will appear in the denominator which is permitted by Maple but not what we want in this situation. To get the plus to appear outside of the fraction, we type the right arrow key (the key with \( \rightarrow \) on it). This moves the cursor out of the fraction back into the baseline of the expression. Then we can enter the + for addition, and another fraction. After we hit the enter key, Maple will simplify the result into a single fraction with any common factors removed from the numerator and denominator.

Now let's do a multiplication. The Maple programming language (like most) uses an asterisk (*) as the symbol for multiplication. However, Maple displays the expression with a centered dot. This may be disconcerting -- what you type is not what you see. When you are in "math mode" Maple will be using fancy typography to display whatever math you are entering. We'll see more of this shortly. See if you can reproduce this result::

\[\begin{array}{|c|c|}
\hline
\text{factorial} & n! \text{ is the product of all the integers between 1 and } n. \text{ It is useful in computations that compute the number of possible ways that something could happen. For example, the number of possible orderings of a deck of playing cards is } 52!. \\
\hline
\end{array}\]

\[\begin{array}{|c|}
\hline
\begin{align*}
2^2 + 1 &= \sqrt{2} \\
2^2 + \frac{4}{5} &= \frac{21}{20} \\
4! &= 24 \\
(3!)! &= 720 \\
52! &= 8065817517094387857166063685 \cdot 6403766975289505440883277 \cdot 82400000000000 \\
\end{align*}
\end{array}\]
We can mix operations. Try to enter and calculate the following:

\[
\frac{1 + \frac{2}{3+4} + 5\cdot6 + 7}{8}
\]

\[
\frac{67}{14}
\]

In order to get that last denominator, we had to select the expression we had entered for the numerator with the mouse, so that the entire contents of the entry rectangle were blue. Then we typed a slash and the denominator appeared beneath it all.

An alternative to using the mouse to enter expression (1.2.1.18) would be to use parentheses. If we type "(1+2/3+4+5*6+7)/8 enter" we will see this:

\[
\frac{\left(1 + \frac{2}{3+4} + 5\cdot6 + 7\right)}{8}
\]

\[
\frac{67}{14}
\]

This allows you to enter complicated expressions without having to use the mouse. Of course, the mouse is still necessary if you want to go back and edit.

We observe in passing that a distinctive feature of Maple is that Maple does exact arithmetic with integers and fractions. It keeps fractions as the ratio of two integers. It will, however, automatically simplify such ratios to lowest terms:

\[
\frac{2}{3} \div \frac{6}{7} = \frac{18}{7}
\]

\[
-2
\]

Making typographical mistakes

Making mistakes is a normal part of using any tool, be it a computer or otherwise. You’ll probably make as many mistakes learning how to use Maple as you would make when learning a new sport, a musical instrument, or when learning how to write a good essay. Some of them will be obvious as soon as you make them, others will be subtle or harder to figure out how to fix.

When you make some kinds of mistakes, the computer may give you an error message. For example, if you make a typo and Maple doesn’t recognize what you enter as being a valid command, it will complain. Here are some typical error messages. At this stage, some of the messages will make sense and you can react appropriately. Others will use vocabulary that is unfamiliar. For those, the best thing to do is to have a clear idea of what you want to enter (by closely imitating examples that are known to work) and checking carefully that what you have typed was accurate.

<table>
<thead>
<tr>
<th>Table 2.6: Examples of Maple error messages</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 +</td>
</tr>
<tr>
<td>Error, invalid sum/difference</td>
</tr>
</tbody>
</table>
We intended to enter "2 + 4" but forgot to type the "4" before we hit *enter (return)*. The appropriate thing to do here is to correct the expression and hit *enter* again.

\[
\begin{array}{c}
2 + 4 \\
\hline
6
\end{array}
\] (2.23)

This time we mistakenly enter the expression with the symbols in the wrong order. Maple complains that it expects some operation to be entered between the two numbers but there isn't one.

\[
\begin{array}{c}
. + 4 \\
\hline
\text{Error, invalid matrix/vector product}
\end{array}
\]

We intended to enter "2+4" but typed a period instead of a 2 by mistake. Even though we are trying to do the same thing, the error message is different because a different symbol (the period) in this context suggests to Maple that we are trying to do linear algebra. The appropriate thing to do here is to correct the expression and hit *enter* again.

\[
\begin{array}{c}
2 + 4 \\
\hline
6 \quad (2.26)
\end{array}
\]

If we ask Maple to do an impossible operation, it sometimes gives an error (depending on the operation). The appropriate question to ask yourself here is "what should I be dividing by instead of zero?".

\[
\begin{array}{c}
\frac{3}{5 + 3} \\
\hline
\text{Error, unable to match delimiters}
\end{array}
\]

We started a sub-expression with a parentheses but forgot to finish it. In Maple, a *delimiter* refers to a parenthesis -- ( or ) -- a bracket [ or ], or a brace { or }. Delimiters are symbols that mark the beginning and end of an expression. In many instances they are necessary to unambiguously indicate meaning. For example, \(5 \cdot (3 + 5)\) evaluates to 40, where as the expression without parentheses \(5 \cdot 3 + 5\) means 20 because multiplications are always done before additions unless the parentheses indicate otherwise.

\[
\begin{array}{c}
\left(3 + \left(5 + \frac{3}{7} \cdot 5\right)\right) \cdot 2 \\
\hline
\text{Error, unable to match delimiters}
\end{array}
\]

This is another instance of the same mistake. We wanted to enter \(\left(3 + \left(5 + \frac{3}{7} \cdot 5\right)\right) \cdot 2\) but misplaced several parentheses.
We intended to enter "1+3" but typed the extra comma in by mistake. We get an error message that talks about sequences, a concept in Maple that we haven't discussed yet (that's coming in the next few chapters). Maple thinks that the sequence is "invalid" because it usually expects commas to be between items, such as "1,2,x".

Maple's language has many other elements in it that we haven't gotten to discuss yet. If you use any of them by mistake then you will see messages with vocabulary we haven't discussed yet.

It's a fairly typical experience for new users to see some messages that you won't be able to gather much intelligence from other than the fact that you made a mistake that you should fix. With knowledge-rich systems such as Maple, you may blunder into sections of the system that you haven't learned yet. The best strategy is to back out of the situation by editing the expression so that it is exactly like something that is known to work.

This one is fairly obvious. In order to fix it though, we need to know what denominator we intended to enter.

**Correcting typographical mistakes**

The standard procedure for fixing a mistake is as you would in a word processor: *edit the mistaken input* and *re-execute the computation*. Here are ways of doing this:

1. Using the mouse, position the cursor where the mistake is. Then use the backspace key to erase the characters you want to get rid of. Type in more characters to replace it.

2. Use the left arrow key (←) to back up. Typing after backing up then inserts the new typing at the point where the cursor was positioned.

3. Use the mouse or other "pointing device" of your computer to select a section of what you typed. New typing then replaces the selection of what you typed.

4. Use the mouse to select a region, then "cut", which you can do through the Maple menu Edit -> Cut. Of course most people use the keyboard shortcut for cutting, which for Windows or Linux is control-X while on the Mac it's command-X.

5. Copying and pasting (control/command-C and control/command-V) also works in Maple.
You may find that sometimes you attempt to create a 2D Math input area but Maple does not compute a result for the input after you hit the enter key. To create a "clickable math" input area if this happens, place the cursor where you want the input area to be, and use the Format->Create Document Block Menu item of the Maple window:

Table 2.7: Create Document Block to force a Math input area wherever the cursor is placed

Exponentiation (powers). Numbers with lots of digits

Use a caret ^ to specify an exponent (a "power"). In math mode, Maple will position the cursor so that the next things you enter will become the exponent. As with fractions, you can get out of the exponent by using the right arrow key → when you want to go back to non-exponent numbers. Try entering these expressions

\[
2^3
\]

\[
\frac{2^{1000} - 2}{2}
\]
We note that Maple does integer and fraction operations exactly. It will not introduce any rounding error into a computation as a calculator would when the answer requires more than ten decimal digits to write down.

There are limits to the number of digits Maple will use for integers or fractions, but they have to do more with ultimate limits of the computer hardware and memory rather than a "pre-ordained" decision about how many digits might be useful to keep. If you type \texttt{kernelopts(maxdigits)} into a Math input area on the worksheet, Maple will print out a number which is the maximum number of digits it can handle in any integer or fraction. On the author's computer,

\[ \texttt{kernelopts(maxdigits)} = 268435448. \]

Note that this is not the value of the maximum number, but how many decimal digits the largest number can have.

For example, Maple can compute the result of

\[ \frac{1}{52!} + \frac{2^{100}}{3^{27}} \]

exactly (try it!).

Exact computation is useful not only for doing algebra but also for things such as computing probabilities through counting, or in deriving mathematical formulas for use in simulation or prediction programs.

**Detecting and fixing vocabulary and "logic" mistakes**

There will be other kinds of mistakes where there is no error message, but the response is not what you want. Sometimes this happens because you say something that while grammatical, means something entirely different from what you meant. This could be as simple as entering "2-3" where you meant to type "2+3". Sometimes it is less obvious, such as mistyping the 23-rd digit of a 55 digit number or the 12th term of a long sum. Or worse yet, it could be because you are using the incorrect vocabulary so what you think you are saying does not have that meaning to the computer.

**Table 2.8: Example of a vocabulary mistake**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x^3 + 5 )</td>
<td>( 2x^3 + 5 )</td>
</tr>
</tbody>
</table>

Suppose we were under the (mistaken) impression could use "x" in Maple to stand for multiplication. We might use the above as a way to calculate "two times three, plus five". However, while there is no error message, what is calculated is not the number we were expecting. The first step towards proceeding to fix this is to realize that it's \textbf{not what we want}. To Maple, this is a legitimate calculation -- you want to create a formula that's "two times the variable x3, plus five".

\[ 2 \cdot 3 + 5 \]

11

(2.33)

Knowing that the proper way to enter multiplication is through a palette, or symbol "*" (asterisk) as explained in

Finally, there are mistakes made because you ask Maple to do the wrong calculation. Even though Maple does what you want, it turns out that what you wanted was wrong!

For example, you may read a word problem and decide to solve the equation \( 3 \cdot x + 2 = 6 \), whose solution is \( x = 4/3 \). But when you plug the solution into the circumstances of the problem, you find that it doesn't solve the word problem. The problem may be because you used faulty logic to decide that \( 3 \cdot x + 2 = 6 \) was the equation, but it was actually \( 2 \cdot x + 4 = 6 \). This is known as an "error in logic" or just a "logic error".

Sometimes it's obvious that you made a mistake because the answer is obviously wrong. For example, seeing "-1" when you are expecting the answer to "2+3" is obvious because you know already that you shouldn't get an answer which is a negative number. Sometimes it isn't so obvious, so you need a way to check the correctness of the result.
2.4 Saving and retrieving your work

You can save your work in a Maple worksheet file through the File -> Save (keyboard shortcut: control/command-S) menu item of the Maple application. A dialog box appears allowing you to type in the name of the worksheet (we've typed in "test" into the box in the .)

The file is saved with a ".mw" suffix (e.g. test1.mw).

To retrieve the file in a subsequent Maple session, start up the Maple application as before and then use the File->Open menu item to initiate the dialog that retrieves the file.

Table 2.9: Maple save menu operation
2.5 Retrieving backups

The "state of Maple" display appears on the bottom left hand side of the Maple window. Usually this display says "Ready", which means that Maple is awaiting your next command. Sometimes it reads "Evaluating..." which indicates that Maple is actively computing an answer. Sometimes it says "Autosaving worksheet", which means that it is saving a copy of the present state of your open worksheets into temporary storage on your computer. The amount of time Maple spends autosaving becomes noticeable in longer Maple sessions when the worksheet contains a lot of results.

Should your computer suddenly lose power or should Maple crash, you can retrieve the last autosaved worksheet by selecting the File -> Recent Documents -> Restore Backup menu item. This will fill your Maple with copies of all the autosaved worksheets. You may then delete them or save them to permanent file space as you wish.

Table 2.10: Maple save dialog box

![Maple save dialog box]

Table 2.11: The Maple state display

![The Maple state display]
2.6 Algebra, plotting and mouse-clickable operations

Algebraic expressions and equations. Solving equations. Working with pieces of expressions.

We don't have to limit ourselves to just numerical calculation in Maple (even if the ability to use as many digits as we wish and exact fractions allows us to do arithmetic more like the math books do). Another distinctive feature of Maple is that you can do algebra by entering expressions with symbols -- the $x, y, z, i,$ and $n$ that we see in algebra books. Maple will automatically collect terms and do some simplifications for us automatically

$$x^2 + 2 \cdot x + 5 + 3 \cdot x$$

$$x^2 + 5x + 5 \quad (2.34)$$

We can even enter equations:

$$\frac{3}{5} \cdot x + 1 = 4 - x$$

$$\frac{3}{5} x + 1 = 4 - x \quad (2.35)$$

$$3 \cdot x + 1 + 4 \cdot x = a \cdot x + b$$

$$7x + 1 = ax + b \quad (2.36)$$

Note that while Maple automatically collected the $x$ terms on the left hand side of the equation, it does not try to do the more interventionist operation of moving all the $x$ terms to the same side of the equation.

Now, enter the following expression, but rather than hitting the enter key after you've entered it, do a right-click. On the Mac, instead of right-clicking hold the control key down then click on the mouse button (this is referred to as "control-click"). A menu of algebraic operations will pop up. Select Factor and see how Maple can factor the polynomial:

$$x^2 + 5 \cdot x - 50 \quad \text{factor} \quad (x + 10)(x - 5)$$

Note that this line does not have a (XX) label for it.

To further demonstrate the right-click (control-click) operations available, enter the following equation. Right click on the expression and a pop-up menu should appear that includes a "solve" item. Select the solve and a submenu will appear where you can specify that you want to solve for $x$.

$$\frac{3}{5} \cdot x + 1 = 4 - x \quad \text{solve for } x \quad \left[ x = \frac{15}{8} \right]$$

For those with previous experience on other systems: some things are different, for a reason

(This section is for people who already know a programming language and are noticing that Maple does some things differently. Other readers can skip this section.)

Most mainstream programming languages work in a similar way. That is why learning Maple has value beyond just being able to use the Maple system -- once you learn Maple, learning Matlab or Java or C is a matter of incremental adjustment. Noticing the differences is educational in that it makes you become aware of the arbitrariness of some of features of languages. If you have used another programming language such as Java or Visual Basic (VB), you will notice that some things do not work the same in Maple as they do there. Some of this can be explained if you know something about the history of programming languages. Some of it is explained by being aware that the designers of Maple had different goals than those who invented Java or VB.
One thing that you have undoubtedly noted is that symbols do not have to have a value associated with them. For example, in Java if you said

\[ k = 5; \]

Then if you were to create another expression in Java such as System.out.println(k^2 + k + k + 3); then "5" would be used as the value of k in the expression and you would end up printing 38. In Maple, you do not have to associate \( k \) with a numerical value before you use \( k \) in an algebraic expression. If there is no prior association, Maple just treats the expression as a formula with symbols in it. It may do some algebraic simplification on what you entered, but it does not need to get a number as a result. Since there was no prior assignment for the expressions in section 2.5.1, the calculation done with them just keeps the formula.

Another thing that is different is that in Maple "=" is used for equations, not assignment.. The operator in Maple corresponding to "=" in Java or VB is "=:" (a colon immediately followed by an equals, with no spaces inbetween). In Maple, if we wanted to associate "5" with the symbol \( k \), then we would do:

\[ k := 5 \]
\[ k^2 + k + 3 + k \]

People who know more than one programming language have a better understanding of which features are change a lot between languages, (such as whether = or := is used for assignment), and which ones are fairly uniform (+ being used for addition, or the use of parentheses in functions and expressions).

Maple does not use "=" for assignment because, being a mathematically oriented language that can handle algebra, it wants to make the entry of equations a natural thing. Its use of "=" for assignment is a feature borrowed from the Algol/Algol 68/Pascal family of programming languages, which picked this operator to make it clear that the assignment operation is different from algebraic equality.

Is "=" better than "="? That's a kind of question that is about as hard to answer as trying to decide whether "uno" is better than "one". If one had a language where you had to do "=-------&%*#*++----" instead of "=" or "=:", you could be critical of the choice because it takes much more effort to enter a 35 character operator than a one or two character one. But the Algol-family choice of "=" has reasonable motivation -- studies of novice programmers have shown that beginners using languages where "=" is the assignment make more mistakes because they confuse its use in mathematics with its use in programming. Novices have been observed to write things like "5=k" which does not work as an assignment, even though mathematically the equations "k=5" and "5=k" mean the same thing.

Just as with architectural design of a building, each feature of a programming language is typically carefully considered. Many features are borrowed or copied from predecessor languages, where they have already been subjected to the test of many people using the feature. If you are not familiar with Algol family languages, you can see examples of them at various educational web sites, such as http://www.engin.umd.umich.edu/CIS/course.des/cis400/algol/average.html and http://portal.acm.org/citation.cfm?id=154766.155365.

**Plotting and approximate numerical solutions**

The right-clickable interface can also activate a plot of a formula you may have entered. Enter this formula, then right-click and select **Plot -> 2d plot**. The automatic defaults for plotting this produce this result.
Table 2.12: Example of Plotting

![Example Plot](image)

Table 2.13: Plot created by right-click -> Plot -> 2DPlot

![Plot Builder](image)

User has clicked on the plot and positioned the cursor at the coordinate (-4.12, 61.60). The cursor was not captured by the screenshot although it is visible under ordinary use.

The 2DPlot operation makes pre-set decisions about the plot, such as the range of \( x \) (-10 to 10), the color of the line, axes labelling, etc. Users can inject their own preferences about these things about this by selecting right-click -> Plot -> PlotBuilder and filling in the dialog box with their choices.
Table 2.14: User-configured plot using PlotBuilder instead of 2DPlot

\[ x^2 - 10 \cdot x + 4 \rightarrow \]

TheExpressionsPaletteandtheCommonSymbolsPalette:enteringTrigs,logs,roots,exponentials

It's possible to get the common functions of high school algebra, pre-calculus, and calculus by using the Expression palette of the Maple Window:

Table 2.15: The Expression palette
For example, to enter the square root of 36, click on the palette entry for $\sqrt{a}$. That expression will appear in the document, with the "a" selected. If you then type 36 on the keyboard, that number will replace the selected text. If you then hit the enter key, Maple will evaluate the expression and produce the exact result "6".

\begin{equation}
x + y + \frac{1}{2} + \frac{1}{4} + \sqrt{36} = x + y + \frac{27}{4}
\end{equation}

You can use the palette multiple times, to create more complicated expressions. Just continue to the use mouse or the arrow keys to move around in the expression. Selecting and typing, backspacing or deleting are all ways of replacing or correcting pieces of the expression.

The Common symbols palette, two panels below the Expression palette, can be used to enter $\pi$ and $e$, the base of the natural logarithm system.

The palette does not have the inverse trig functions, so you have to enter them through typing. Their names are: arcsin, arccos, arctan, etc.
Table 2.16: Examples of palette-driven computation

\[
\sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = \sqrt{2} \\
\left(\frac{\sqrt{1024} + \ln\left(\frac{3}{2}\right)}{13}\right) \cdot \pi = \frac{98}{39} \pi \\
\arcsin\left(\sin\left(\frac{1}{4}\pi\right)\right) = \frac{1}{4} \pi
\]

(2.40) (2.41) (2.42)

Approximate numerical (calculator-type) arithmetic in Maple

If you enter expressions with integers, exact fractions, and symbols such as π and e, then Maple will perform exact calculations rather than give approximate answers as a conventional calculator would. You can get approximations by selecting the "numerically solve" instead of the "solve" option from the right-click pop-up menu.

Table 2.17: Examples of computing with approximate solving

<table>
<thead>
<tr>
<th>Expression</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^2 - 10 \cdot x + 4 = x^2 - 10 \cdot x + 4)</td>
<td>(x = 5 + \sqrt{21}, \ x = 5 - \sqrt{21})</td>
</tr>
<tr>
<td>(x^2 - 10 \cdot x + 4)</td>
<td>0.4174243050, 9.582575695</td>
</tr>
</tbody>
</table>

If you have an exact expression, you can ask Maple to approximate it to 5, 10, 20, or more digits. In this mode, Maple can be used as a super-accurate calculator.

Examples of numerical computation

1. Enter fraction, select approximate->20 from right-click pop-up menu.

\[\frac{47}{52} + \frac{4}{3}\] at 20 digits \(\rightarrow 2.2371794871794871795\)

2. Enter exact expression, select approximate->5 from right click pop-up menu

\[\sin\left(\frac{\pi}{10}\right)\] at 5 digits \(\rightarrow 0.30902\)

3. Enter equation. Then solve->solve, then select Element->1, then right hand side, then approximate->10
Evaluation, and selection of pieces.

Sometimes you wish to evaluate an expression for a particular value of a variable. There is a right-click operation that does this.

Table 2.18: Evaluate at a point

<table>
<thead>
<tr>
<th>Expression</th>
<th>Operation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 - 2 \cdot a \cdot x = 0 )</td>
<td>evaluate at point</td>
<td>( \frac{1}{4} - a = 0 )</td>
</tr>
<tr>
<td>( x^2 - 2 \cdot a \cdot x = 0 )</td>
<td>evaluate at point</td>
<td>( x^2 - 6y^2 \cdot x = 0 )</td>
</tr>
<tr>
<td>3\cdot y + 5</td>
<td>evaluate at point</td>
<td>14</td>
</tr>
</tbody>
</table>

This operation will give a pop-up menu that will allow us to choose values for all the variables. In the first example, we picked \( \frac{1}{2} \) for a value of \( x \). Note that the pop-up menu will show what you typed rather than displaying 2D math. In the second example, we specified "3*y^2" as the value for \( a \). In the third example, we picked 3 as the value for \( y \).

Using the right-click menu, it's possible to select or extract a portion of an expression for further work.

Table 2.19: Operations on equations, multi-part expressions

<table>
<thead>
<tr>
<th>Expression</th>
<th>Operation</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x = \frac{\sin(a)}{r^2 - 1} )</td>
<td>right hand side</td>
<td>( \frac{\sin(a)}{r^2 - 1} )</td>
</tr>
<tr>
<td>( x = \frac{\sin(a)}{r^2 - 1} )</td>
<td>left hand side</td>
<td>( x )</td>
</tr>
</tbody>
</table>

One of the options in the right-click menu is "right hand side". It only works for equations.

Solving this quadratic equation reveals that there are two solutions. Right-clicking on the solution and then selecting entry 1 gives the first solution, enclosed in brackets \([ \ \]\). Right-clicking on that and again selecting entry 1 gives the first solution, without the brackets. Right-clicking on that and selecting approximation gives a calculator approximation to the root.

2.7 A quick-reference summary to this chapter

<table>
<thead>
<tr>
<th>Example</th>
<th>Description</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic</td>
<td>Use +, *, -, /, ^ for arithmetic. Hitting the Enter key produces a labelled result.</td>
<td>2 D Math input mode displays the textbook-like version of what you input. Maple's simplification automatically combined fractions and places things in lowest terms.</td>
</tr>
<tr>
<td>( 2 + \frac{3^2}{4} - \frac{1}{6} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{49}{12} )</td>
<td>(2.43)</td>
<td></td>
</tr>
<tr>
<td>5!</td>
<td>Use ! for factorial</td>
<td>Do you know what 5!! (double factorial) means?</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Making mistakes</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\[
2 + \left( \frac{3}{5} \right)
\]

Error, unable to match delimiters

Error message mistakes (from typos or mistakes in intensions)
The easiest ones to detect. You have to figure out what you are doing wrong, though. The error message may not always be helpful in advising you on this, although it often is.

A farmer plants a fence post every foot, for 1250 feet. At that point, he switches to planting the fence posts every 4.7 feet for another 940 feet. How many fence posts does he need in all?

\[
\frac{1250}{1} + \frac{940}{4.7}
\]

1450.000000

(2.45)

Maple did do the arithmetic in the above calculation correctly. The problem is that it's the wrong calculation. Do you see how to get the right answer?

"Logic errors" You are asking Maple to compute something that it understands, so it gives you an answer. However, this answer doesn't really solve your problem. You need to find a more appropriate computation, which you can only do by thinking about whether you are asking the computer to do something different from what is needed.

Often you can find these kinds of mistakes by looking at simpler versions of the problem where the answer can be figured out with paper and pencil. Then you can "scale up" the answer to handle the actual problem you have.

The correct answer is 1251 + 201 1452 fence posts. The computer did what it was asked to do -- the problem was that it was asked to do the wrong thing.

**Editing (fixing mistakes)**

<table>
<thead>
<tr>
<th>Action</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>backspace, delete erase starting from current cursor selection</td>
<td></td>
</tr>
<tr>
<td>Arrow keys→← move cursor within current selection</td>
<td></td>
</tr>
<tr>
<td>Select with mouse/type replaces selected text</td>
<td></td>
</tr>
<tr>
<td>Cut, copy and paste of a selection works as it does with a text processor</td>
<td></td>
</tr>
</tbody>
</table>

**File saves, opens**

Save files with **File -> Save** or **File -> Save As**. Open a saved file with **File -> Open**. Other File operations are similar to that of standard word processors.

**Functions and math symbols**

\[
\sqrt[3]{\text{csc} \left( \frac{\pi}{2} \right)} + e^{(1 + e)^{1/3}}
\]

(2.46)

Insert math into an expression by using the Expression Palette. You can enter \( \pi \) using the Common Symbols Palette. \( e \) (the natural logarithm base) can also be entered this way. Note: typing e from the keyboard does not enter this symbol.

Chapter 2 demonstrated the following functions and symbols:

<table>
<thead>
<tr>
<th>Square roots, ( n )-th roots</th>
<th>natural logarithms (base ( e ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trig functions: sin, cos (trig functions all use radians, not degrees)</td>
<td>Base 10 logarithms</td>
</tr>
<tr>
<td>arcsin, arccos, arctan sec, csc, \pi, e</td>
<td></td>
</tr>
</tbody>
</table>

\[
\ln(e^2 - \sqrt{e}) \rightarrow simplify \rightarrow \frac{5}{2}
\]

If you are entering a function by the keyboard rather than the palette, you must enclose the function's argument in parentheses.

**Algebra**

\[
x^2 - 2x - 15 = 0 \quad \text{(left hand side)}
\]

Right-click (control-click on Mac) on an entered expression to get the pop-up menu.

Chapter 2 demonstrated examples of the following operations:
<table>
<thead>
<tr>
<th>Expression</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin(x) = 1 )</td>
<td>( \text{solve numerically} \rightarrow x = \frac{1}{2} \pi )</td>
</tr>
<tr>
<td>( \sin(x) = 1 )</td>
<td>( \text{solve numerically} \rightarrow 1.570796327 )</td>
</tr>
</tbody>
</table>
| \( x^2 - \arcsin(x) \) | \( \text{select (n-th part) of an expression} \) |}

### Plotting

#### Plots→2d plot

The expression must be something that involves a single variable and will result in a number when a value is used for that variable. Thus you can’t do a 2d plot of \( x^2 - a \) because you wouldn't get a number if you picked a value just for \( x \) (or just for \( a \)).

Maple uses defaults for the plot range, and the plot color.

Trying to plot an equation produces an implicit plot (see next appendix).

#### Plots→plot builder → 2d plot

A dialog box appears that allows you the select from many more options, such as plot color, the line style, vertical and horizontal ranges, captions, etc.

### Limited precision (decimal point) numbers

<table>
<thead>
<tr>
<th>Expression</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \cos(x^2) = \sqrt{x} )</td>
<td>( \text{solve} \rightarrow 0.7352027350 )</td>
</tr>
</tbody>
</table>

Exact numbers in Maple have no decimal points.

Use of limited precision numbers in algebra (e.g. factoring, differentiation, solving) may not produce good results.
Symbolic constants such as π and e entered from the Common Symbols Palette are also exact.

Numbers with decimal points in Maple cause arithmetic calculations to be done approximately.

solve->numerically solve produces approximate solutions.

.right-click->approximate->n takes an exact numerical expression and approximates it.

Use them in Maple only when an approximate result is desired.

Numbers like .25 or .6015 are limited precision. If you want exact algebra done, use 1/4 or 6015/10000, etc.

In very large calculations, limited precision calculations may be noticeably faster than those with exact arithmetic. Most of the time there isn’t an appreciable difference.

<table>
<thead>
<tr>
<th>Evaluate at a point</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 - 2a\cdot x = 0 )</td>
<td>evaluate at point ( \frac{1}{4} - a = 0 )</td>
</tr>
<tr>
<td>( x^2 - 2a\cdot x = 0 )</td>
<td>evaluate at point</td>
</tr>
<tr>
<td>( x^2 - 6y^2\cdot x = 0 )</td>
<td>evaluate at point ( 14 )</td>
</tr>
</tbody>
</table>

This operation will give a pop-up menu that will allow us to choose values for all the variables. In the first example, we we picked \( \frac{1}{2} \) for a value of \( x \). Note that the pop-up menu will show what you typed rather than displaying 2D math. In the second example, we specified "3*y/2" as the value for \( a \). In the third example, we picked 3 as the value for \( y \).

<table>
<thead>
<tr>
<th>Operations on equations</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>right hand side, left hand side</td>
<td>( x = \frac{\sin(a)}{t^2 - 1} ) right hand side ( \frac{\sin(a)}{t^2 - 1} ) ( x = \frac{\sin(a)}{t^2 - 1} ) left hand side ( x )</td>
</tr>
</tbody>
</table>

One of the options in the right-click menu is "right hand side". It only works for equations.

<table>
<thead>
<tr>
<th>Operations on multi-part expressions</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>select entry ( x^2 - 4\cdot x = 4 )</td>
<td>solve for ( x )</td>
</tr>
<tr>
<td>( [[x = 2 + 2\sqrt{2}], [x = 2 - 2\sqrt{2}]] )</td>
<td>select entry 1 ( x = 2 + 2\sqrt{2} ) at 5 digits ( x = 4.8284 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Operations on symbolic expressions</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>solve-&gt;solve ( x^2 - 1 )</td>
<td>( {x = 1}, {x = -1} )</td>
</tr>
<tr>
<td>solve-&gt;solve for a variable ( x^2 - 2a\cdot x = 0 )</td>
<td>solve for ( x ) ( {x = 0}, {x = 2a} )</td>
</tr>
<tr>
<td>solve-&gt;numerically solve ( x = \cos(x) )</td>
<td>solve ( 0.7390851332 )</td>
</tr>
</tbody>
</table>

The thing to try when there is a numerical answer but the exact solution is too complicated to understand or Maple can't find an exact solution.

Factoring \( x^2 - 1 \) \( (x - 1)\cdot(x + 1) \) \( \cos(x)^2 - \sin(x)^2 \) \( \cos(x) - \sin(x) \) \( (\cos(x) + \sin(x))\cdot(\cos(x) + \sin(x)) \)

Factoring can simplify an expression sometimes. Factoring doesn't know the trig simplification rules, though.
The expression must be something that involves a single variable and will result in a number when a value is used for that variable. Thus you can't do a 2d plot of \( x^2 - a \) because you wouldn't get a number if you picked a value just for \( x \) (or just for \( a \)).

Maple uses defaults for the plot range, and the plot color.

A dialog box appears that allows you the select from many more options, such as plot color, the line style, vertical and horizontal ranges, captions, etc.

<table>
<thead>
<tr>
<th>Operations on equations</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>right hand side, left hand side</td>
<td>( x = \frac{\sin(a)}{g^2 - 1} ) right hand side ( \rightarrow \frac{\sin(a)}{g^2 - 1} )</td>
</tr>
<tr>
<td></td>
<td>( x = \frac{\sin(a)}{g^2 - 1} ) left hand side ( \rightarrow ) ( x )</td>
</tr>
</tbody>
</table>
This moves the entire side of an equation to the other side.

<table>
<thead>
<tr>
<th>Operations on constant expressions</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>approximate-&gt;5 (or 10, 20, 50)</td>
<td>$\tan\left(\frac{\pi}{10}\right) \sqrt{\frac{1}{10}}$ at 20 digits $0.34157868529293212152$</td>
</tr>
<tr>
<td>$x = \ln(5000!)$ at 20 digits</td>
<td>$x = 37591.143508876766569$</td>
</tr>
</tbody>
</table>

Uses approximation methods to turn constants into an approximate (decimal point) number with 5 (10, 20, 50) digits.
3 Chapter 3 Technical word processing

3.1 Chapter Overview

We learn how to use Maple as a word processor. This allows us to "write up" reports, combining technical writing with math formulae, calculated results, pictures, tables, etc. Many of the features are highly similar to Microsoft Word or similar WYSIWYG (what you see is what you get) word processors. The strength of Maple's word processing is that it makes it easy to enter technical formulae, and that the word processing and calculation can be done in the same document.

3.2 Maple as a word processor

Maple documents allow a mixture of text and mathematics. By default, Maple expects that when you position the cursor by clicking somewhere in the document, you will be entering math and be wanting it do to a calculation. The document is in what is called math entry mode.

Table 3.1: Maple in math entry mode

![Math button on Maple toolbar]

You can tell whether the document is in math entry mode because the Math button on the Maple toolbar will be gray, and the "C" menu item says 2D Math.

The other mode of operation for Maple documents is text mode. When in text mode, Maple has the behavior of a word processor. It just shows what you typed. Hitting enter while you are in text mode just causes text entry to move to the next line. It does not cause any calculation to be done with what you typed.
You can switch to entering text in the following way:

1. Position the cursor at the spot where you want to enter text.

2. Click on the **Text** button on the Maple toolbar. This places the Maple document in **text entry mode**. Alternatively you can switch to Text mode by typing control-T (on Macintosh, command-T) or by using the Maple menu bar Insert->Text. You can tell when you've switched to text entry mode because the Text button will be gray, and the "C" menu item says **Text**.

3. With the keyboard, enter your verbiage. When you are in text mode, you will also see that the menu bar will be enabled for boldface, italics, underlining, left-/center-/right- justified text, colored text, and colored backgrounds, and bulleted/numbered text.

4. To switch back to math, click on the **Math** button on the Maple toolbar. Alternatively you can type control-R (on Macintosh, command-R) use the Maple menu Insert->2-D Math.

### Table 3.2: Document after control-T (or Insert->Text)

A Maple worksheet in text mode in OS X. Although it is hard to see, the cursor is positioned at top left of screen.

You can do mathematical word processing without any computation by switching between text and math modes, using the Palettes to help you enter the math. As long as you don't hit the return (enter) key, the math will not cause any calculation.
Table 3.3: Document with a mixture of text and math

Richard saw in his physics textbook, *Stephen Hawking for Dummies*, a description of Newton's law of gravitation:

\[ F = \frac{G \cdot m_1 \cdot m_2}{R^2} \]

where it was expected that \( m_1 \), \( m_2 \), and \( R \neq 0 \). Although he didn’t consider himself a strong physics student, he was glad that hadn’t dumbed down the material so much that it lost all the mathematics.

It is possible to mix text and the results of calculations in a paragraph. Typing control-= (command-=) when the cursor is in a math expression will cause Maple to print an "=" and then the result of evaluating the expression on the same line. This is an alternative to hitting the enter key and allows those kinds of calculations to be mixed with text.

Table 3.4: control-= puts the results of a calculation in the midst of text

With a loan of $250,000 and an interest for 3.5% annual interest, the amount of interest after one year would be $250000 \cdot 1.035 = 2.58750000 \cdot 10^5$ dollars.
3.3 Shortcuts to entering math symbols

Using the Palettes, we can enter a wide variety of mathematics -- expressions, math symbols, Greek letters (using the Greek Palette), arrows, etc. There are additional Palettes not shown by default, which you can get by View → Palettes → Show All Palettes. However, you can enter many symbols in math mode from the keyboard through "shortcuts". Most of the shortcuts consists of typing the textual name of the symbol or some abbreviation of it, and then hitting the escape key -- the key labelled Esc on many keyboards.

For example, to enter the symbol $\infty$ while in math mode, you can type infin and then hit the escape key. A pop-up menu of choices will appear to allow you to complete entry of the symbol. With practice, this can be a faster way of entering "infinity" than using the Palettes.

Table 3.5: Keyboard shortcuts in math mode through the escape key

<table>
<thead>
<tr>
<th>Description</th>
<th>Image</th>
</tr>
</thead>
<tbody>
<tr>
<td>In math mode, we type infin.</td>
<td><img src="image1" alt="Image" /></td>
</tr>
<tr>
<td>After hitting the escape key, a menu of completions appears.</td>
<td><img src="image2" alt="Image" /></td>
</tr>
<tr>
<td>We pick the first alternative (either by hitting the return key or by operating the mouse to select the first option) and what we typed is replaced by the selection.</td>
<td><img src="image3" alt="Image" /></td>
</tr>
</tbody>
</table>

Greek symbols can be entered by typing the romanized name of the letter, followed by escape. For example, in math mode, typing omega followed by escape produces $\omega$. Typing Omega followed by escape produces $\Omega$ (the upper case version of the Greek letter).

A shortcut to entering the symbolic constant $e$ (the base of the natural logarithm) is to type e, then hit the escape key, then return.

You can see a summary to all of the conveniences Maple offers through Help → Quick Reference.

3.4 Other word processor features

Inspection of the worksheet toolbar reveals many more word processing features: line justification, bold face and italics, numbered items, colored letters or backgrounds, font sizes, and font types. The Insert operation on the Maple toolbar allows creation of Tables and Images (graphics files). Rudimentary drawings can be inserted through Insert → Canvas. We encourage you to explore and make use of the features on your own.
3.5 Troubleshooting word processing

A phenomenon that you may encounter is *not being able to switch back to math mode from text mode*, even after performing the operation that should do so (clicking on the Text button of the document toolbar, typing `control-T`, performing Insert → 2DMath, etc. This may be due to the worksheet losing track of where you are in the document. A "sure-fire" cure for switching modes is to position the cursor at the point where you want to enter math, then do Format → Create Document Block. A dashed box will appear at the location of the cursor, indicating that it is again in math mode.

**Tools->Spellcheck** (alternatively, the F7 key) will run a spelling check on the non-math part of your document.

### 3.6 Summary of Chapter 3 material

<table>
<thead>
<tr>
<th>Name</th>
<th>Menu operation</th>
<th>Key short cut</th>
</tr>
</thead>
</table>
| Important word processing operations in a Maple worksheet | Insert→2D Math  
Click on "Math" oval in menu bar just below names of worksheets. | control-R (command-R on Mac) |
| Switch entry to 2D Math mode        | Insert→Text  
Click on "Text" oval in menu bar just below names of worksheets. | control-T (command-T) |
| Switch entry to Text mode           | Use a keyboard shortcut in Math mode | Type the shortcut, then hit the *escape* key. For example, typing omega and then *escape* will turn the text into $\omega$. Typing e and then *escape* will allow you to turn the text into the symbolic constant $e$ without needing the Expression Palette. |
4 Chapter 4 Assignment

4.1 Chapter Overview

We learn how to label results with symbolic names through the operation of "assign to a name", sometimes called assignment. This allows us to reuse the results in subsequent steps of a multi-step calculation without retyping it.

The keyboard operation := provides a keyboard shortcut for assignment. := will be used heavily in later work in programming as we shift from mouse/menu operation to textual specification of calculations.

4.2 Assignment: remembering results for future use

We can compute a result and label it with name. This action is called assignment. We can do this with the clickable menu by the action right-click (control-click on Macintosh) → assign to a name. A pop-up menu will appear asking us to fill in the name that we want to use.

Once we have assigned a result to a name, we can use the name, and Maple will use the assigned value.

A name can be any sequence of upper- or lower-case letters, digits and the underscore character _ . It must start with a letter. Maple distinguishes between upper and lower case letters, so result and Result are considered different names.

In programming, the term variable is used interchangeably with name. Both refer to an identifier which the action of assignment associates with a computed result. Computer books often talk about "assigning the result to a variable" which means the same thing as "assigning the result to a name".

Table 4.1: Assignment

<table>
<thead>
<tr>
<th>expression</th>
<th>name</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin\left(\frac{\pi}{4}\right) + 1 )</td>
<td>trigResult</td>
<td>After computing a result, we assign it to the name trigResult by clicking on the result, selecting the menu item assign to a name and then typing in the name.</td>
</tr>
<tr>
<td>( \frac{1}{2} \sqrt{2} + 1 )</td>
<td></td>
<td>(4.1)</td>
</tr>
<tr>
<td>assign to a name</td>
<td>trigResult</td>
<td>(4.2)</td>
</tr>
<tr>
<td>( (\text{trigResult} + 1) \cdot (\text{trigResult} - 1) )</td>
<td></td>
<td>We can then use the name instead of repeatedly entering or copying expressions.</td>
</tr>
<tr>
<td>( \frac{1}{2} \left( \frac{1}{2} \sqrt{2} + 2 \right) \sqrt{2} )</td>
<td></td>
<td>(4.3)</td>
</tr>
<tr>
<td>at 5 digits</td>
<td></td>
<td>(4.4)</td>
</tr>
<tr>
<td>1.9142</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
4.3 Assignment keyboard shortcut :=

The Maple operator := (colon immediately followed by an equals) also performs assignment.

The general form of the assignment operation when using the keyboard is

<table>
<thead>
<tr>
<th>Form</th>
<th>Examples</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>name := expression whose value will be assigned</td>
<td>x := 5</td>
<td>Assigns the name x the value 5.</td>
</tr>
<tr>
<td>poly := ( z + \frac{5}{2} + \sin\left(\frac{\pi}{3}\right) )</td>
<td>Assigns the name poly the value consisting of ( \frac{5}{2} + \sin\left(\frac{\pi}{3}\right) ) + whatever the assigned value of z is. If no value has been assigned the name z, then the result is the algebraic formula: ( z + \frac{5}{2} + \sin\left(\frac{\pi}{3}\right) ).</td>
<td></td>
</tr>
</tbody>
</table>

When you enter an expression in Math mode (even if it's just a name, without any arithmetic), you ask Maple to evaluate what you have entered. Maple also (before it does anything else) figures out the values assigned to names that appear in the expression. Then it does arithmetic, function calculation, and any other operations you've described in the expression.

If you use a name/variable in an expression, and it has no assigned value, then Maple uses the rule that the value of an name with no assigned value is just the name itself.

Table 4.2: Assignment

<table>
<thead>
<tr>
<th>Examples of assignment with :=</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p := x^2 + x + ax + 5 )</td>
<td>We assign the name p the value of the expression. Note that since x and a have not been assigned values, the results of evaluation just leaves them as symbols.</td>
</tr>
<tr>
<td>( x^2 + x + ax + 5 )</td>
<td>(4.5)</td>
</tr>
<tr>
<td>( p + 1 )</td>
<td>If we enter an expression containing p, its value is plugged in for the calculation of the result.</td>
</tr>
<tr>
<td>( x^2 + x + ax + 6 )</td>
<td>(4.6)</td>
</tr>
<tr>
<td>( x := 3 )</td>
<td>Here we assign the name x the value 3.</td>
</tr>
<tr>
<td>( 3 )</td>
<td>(4.7)</td>
</tr>
<tr>
<td>( p )</td>
<td>If we now do a calculation with p, the value of x is used since p's value mentions x. There may be a chain of assignments that Maple must look at to evaluate an expression.</td>
</tr>
<tr>
<td>( 17 + 3a )</td>
<td>(4.8)</td>
</tr>
<tr>
<td>solve ( a = \frac{17}{3} )</td>
<td>We can solve the result 1.3.4 for a by right clicking that expression.</td>
</tr>
<tr>
<td>( a = \frac{17}{3} )</td>
<td>(4.9)</td>
</tr>
<tr>
<td>( x := 4 )</td>
<td>We change the value of x by assigning it a different value.</td>
</tr>
<tr>
<td>( 4 )</td>
<td>(4.10)</td>
</tr>
<tr>
<td>( p )</td>
<td>When we do another calculation with p, the most recent assigned value of x is used.</td>
</tr>
<tr>
<td>( 25 + 4a )</td>
<td>(4.11)</td>
</tr>
</tbody>
</table>
### Examples of assignment with `:=`

<table>
<thead>
<tr>
<th>Equation</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y := x )</td>
<td>This a way of assigning 4 -- the current value of ( x ) to the name ( y ).</td>
</tr>
<tr>
<td>( y := x )</td>
<td>4</td>
</tr>
</tbody>
</table>

\[ (4.13) \]

<table>
<thead>
<tr>
<th>Equation</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>We can undo the connection between ( x ) and any value by unassigning ( x ). This operation produces no output, so no label. We can barely tell that it has happened. The quote marks surrounding the ( x ) -- ' ( x ) ' are mandatory, otherwise ( x ) would be replaced by its value and Maple would try to unassign the corresponding value rather than ( x ) itself.</td>
</tr>
<tr>
<td>( p )</td>
<td>( x^2 + x + ax + 5 )</td>
</tr>
</tbody>
</table>

\[ (4.14) \]

<table>
<thead>
<tr>
<th>Equation</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>We may have unassigned ( x ), but ( y )'s unassigned value -- 4 -- is unaffected. The assignment just connects the name and the value determined when the assignment was performed (back at ( (4.13) )). The information about which variables or expressions were used to figure out what the value was is not retained.</td>
</tr>
<tr>
<td>( y )</td>
<td>4</td>
</tr>
</tbody>
</table>

\[ (4.15) \]

---

### 4.4 How to think about assignment: a mental model

The operation of assignment uses part of the computer's memory to remember the association of the name with the result. A useful mental model of assignment is to think of the computer creating a memory slot containing the result, labeled with the name being assigned to.

Table 4.3: Mental model of assignment

<table>
<thead>
<tr>
<th>Command</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p := x^2 + x + ax + 5 )</td>
<td>The mental model has a slot labeled ( p ) with the value ( x^2 + x + ax + 5 ):</td>
</tr>
<tr>
<td>( x := 3 )</td>
<td>If we then assign ( x ) a value, then there are two slots created. The second assignment only works on the slot associated with ( x ).</td>
</tr>
</tbody>
</table>

\[ (4.16) \]  
\[ (4.17) \]
If we assign $p$ another value, then the memory slot associated with $p$ is cleared out and replaced with the new value. Note that the act of computing the new value for $p$ causes Maple to use any assigned values for $x$ that currently exist. This is different than with calculation (1.2.5). At that time, $x$ had no assigned value.

If we assign $p$ another value, then the memory slot associated with $p$ is cleared out and replaced with the new value. Note that the act of computing the new value for $p$ causes Maple to use any assigned values for $x$ that currently exist. This is different than with calculation (1.2.5). At that time, $x$ had no assigned value.

If we unassign a variable, then we can think of the slot as being deleted from the computer's memory. Unassigning $x$ does not unassign $p$ or change $p$'s assigned value.
4.5 The state of the Maple session and the look of the worksheet

When you first start up Maple with a blank worksheet, you haven't done any assignments. Thus it isn't surprising that a blank worksheet has no assigned variables. Using the mental model of the previous section, Maple has not allocated any memory to remember things -- no slots, no associations between results and names.

The state of a Maple session consists of all the variables that are currently assigned, and what their values are. The mental model of assignments is exactly the state of the session. The state changes every time we do another assignment or unassignment.

Only an assignment operation can change the state of the session. We know that we can jump back and execute a line in the worksheet a second time, just by positioning the cursor there and hitting enter (return). This raises the possibility that the way the worksheet looks is not an accurate reflection of the state of the session -- variables may have different values than what you'd think from reading the worksheet from top to bottom.

If you are not sure what the current value of a variable is, you can find out what it is by entering the name of the variable and hitting enter.

A saved worksheet does not save the state of the session (the variable assignments). Opening a saved worksheet file does not cause it to automatically execute the operations in the worksheet. This gives you a chance to edit the worksheet and possibly change the calculations specified, before carrying out the instructions.

Table 4.4: Example: The state of a Maple Session

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p := x^2 + x + a \cdot x + 5$</td>
<td>When we first load a worksheet, the state of the session is blank. For instance, we may see an equation like this in our worksheet, where it appears that the variable $p$ stores the function $x^2 + x + a \cdot x + 5$. However, the variable $p$ does not store anything yet!</td>
</tr>
<tr>
<td></td>
<td>(4.20)</td>
</tr>
<tr>
<td>$y := 2 \cdot p$</td>
<td>This is evident if we immediately try to use the function $p$ in another equation. Although we might expect $y$ to equal $2 \cdot x^2 + 2 \cdot x + 2 \cdot a \cdot x + 10$, it actually equals $2 \cdot p$. This is because we have just loaded the worksheet, and $p$ is currently unassigned.</td>
</tr>
<tr>
<td></td>
<td>(4.21)</td>
</tr>
<tr>
<td>$p := x^2 + x + a \cdot x + 5$</td>
<td>To make $y$ equal to $2 \cdot x^2 + 2 \cdot x + 2 \cdot a \cdot x + 10$, we need to go back to the line where we assigned $p$ and hit enter. Then we can go back to where we assigned $y$ and hit enter to get the expected result.</td>
</tr>
<tr>
<td></td>
<td>(4.22)</td>
</tr>
<tr>
<td>$y := 2 \cdot p$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(4.23)</td>
</tr>
</tbody>
</table>
4.6 restart causes all assignments to be forgotten

We've seen that it's possible to erase a particular assignment using unassign. If we want to forget all assignments we've made so far, then we can use restart. This can be useful in situations where you've done some work and made some assignments, but now want to switch to working on a different problem and would like Maple to forget about the assignments you made before. It is generally a good idea to restart at the beginning of unrelated sections just in case variables were previously assigned values that might not be related to their use in the new section.

Table 4.5: Evaluation of expressions involving assigned variables

<table>
<thead>
<tr>
<th>Examples of assignment with :=</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p := x^2 + x + a \cdot x + 5$</td>
<td>We were expecting p's value to be the expression $x^2 + x + a \cdot x + 5$. Evidently we must have done some assignment to x previous in the session that's &quot;contaminating&quot; this operation. If we had an accurate mental model of the state of session, it would have included an assignment of 5 to x. We would like to forget about that assignment and any others we've done previously in the session.</td>
</tr>
<tr>
<td>$p := x^2 + x + a \cdot x + 5$</td>
<td>This has the effect of wiping out all assignments from the state of Maple session. The mental model has no assignments in it at this point.</td>
</tr>
<tr>
<td>$p := x^2 + x + a \cdot x + 5$</td>
<td>Now we have the desired result.</td>
</tr>
</tbody>
</table>

4.7 Evaluation and assignment

Assignment really requires two steps. The first is figuring out the result. The second is assigning the result to the name. The "figuring out the result" step is called evaluation.

Evaluation in Maple proceeds in two phases. The first is to determine if any of the symbols in the expression being evaluated have an assigned value. If so, those values are used. If those values involve other symbols, those are in turn checked for values, etc.

Symbols without an assigned value have their own names as their value. This allows you to enter an expression such as $x^2 + 2 \cdot x + 5$ in x and use the x's as symbols in the normal mathematical style as long as you don't assign x a value.

Table 4.6: Evaluation of expressions involving assigned variables

<table>
<thead>
<tr>
<th>Examples of assignment with :=</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>restart</td>
<td>We start fresh through restart.</td>
</tr>
<tr>
<td>$p := x^2 + x + a \cdot x + 5$</td>
<td>We assign the name p the value of the expression. Since x and a have not been assigned values, the results of evaluation just leaves them as symbols. The mental model of the state of the Maple session just has one assignment.</td>
</tr>
<tr>
<td>$x := 3$</td>
<td>If we assign x a value, then evaluating p causes its assigned value $x^2 + x + a \cdot x + 5$ to be evaluated.</td>
</tr>
<tr>
<td>$p$</td>
<td>$17 + 3a$</td>
</tr>
</tbody>
</table>

(4.26) (4.27) (4.28)
### Examples of assignment with `:=`

<table>
<thead>
<tr>
<th>Expression</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>&gt; x := 4</code></td>
<td>If we change the assigned value of <code>x</code> and evaluate <code>p</code> again, this time 4 is used everywhere in the expression <code>x^2 + x + a*x + 5</code>. At this point, the mental model of the state of the Maple session is:</td>
</tr>
<tr>
<td><code>output redirected..</code></td>
<td></td>
</tr>
<tr>
<td><code>4</code></td>
<td>(4.29)</td>
</tr>
<tr>
<td><code>&gt; p</code></td>
<td></td>
</tr>
<tr>
<td><code>output redirected..</code></td>
<td></td>
</tr>
<tr>
<td><code>25 + 4 a</code></td>
<td>(4.30)</td>
</tr>
<tr>
<td><code>&gt; a := y</code></td>
<td>Assigning a a value also changes the result of evaluating p. The assigned value of p hasn't changed, but the result of evaluating p takes into account that a now has a value. After (1.7.12) the mental model of the state of the Maple session is:</td>
</tr>
<tr>
<td><code>output redirected..</code></td>
<td></td>
</tr>
<tr>
<td><code>y</code></td>
<td>(4.31)</td>
</tr>
<tr>
<td><code>p</code></td>
<td></td>
</tr>
<tr>
<td><code>25 + 4 y</code></td>
<td>(4.32)</td>
</tr>
<tr>
<td><code>&gt; result2 := p</code></td>
<td>Changing the value of <code>a</code> causes a different result when evaluating <code>p</code> again, but doesn't change the result of evaluating <code>result2</code>. This is because <code>result2</code> does not change with the assignment to <code>a</code> done in (4.34). Its value is still <code>25 + 4 y</code>. The mental model of the state of the session after the operations (4.26) through (4.36) are done is:</td>
</tr>
<tr>
<td><code>output redirected..</code></td>
<td></td>
</tr>
<tr>
<td><code>25 + 4 y</code></td>
<td>(4.33)</td>
</tr>
<tr>
<td><code>&gt; a := z + 1</code></td>
<td></td>
</tr>
<tr>
<td><code>output redirected..</code></td>
<td></td>
</tr>
<tr>
<td><code>z + 1</code></td>
<td>(4.34)</td>
</tr>
<tr>
<td><code>&gt; p</code></td>
<td></td>
</tr>
<tr>
<td><code>output redirected..</code></td>
<td></td>
</tr>
<tr>
<td><code>29 + 4 z</code></td>
<td>(4.35)</td>
</tr>
<tr>
<td><code>&gt; result2</code></td>
<td></td>
</tr>
<tr>
<td><code>output redirected..</code></td>
<td></td>
</tr>
<tr>
<td><code>25 + 4 y</code></td>
<td>(4.36)</td>
</tr>
</tbody>
</table>

---

### Commentary

- **If we change the assigned value of `x` and evaluate `p` again, this time 4 is used everywhere in the expression `x^2 + x + a*x + 5`. At this point, the mental model of the state of the Maple session is:**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>x</code></td>
<td>4</td>
</tr>
</tbody>
</table>

- **Assigning a a value also changes the result of evaluating p. The assigned value of p hasn't changed, but the result of evaluating p takes into account that a now has a value. After (1.7.12) the mental model of the state of the Maple session is:**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>p</code></td>
<td><code>x^2 + x + a*x + 5</code></td>
</tr>
<tr>
<td><code>a</code></td>
<td><code>y</code></td>
</tr>
</tbody>
</table>

- **Changing the value of `a` causes a different result when evaluating `p` again, but doesn't change the result of evaluating `result2`. This is because `result2` does not change with the assignment to `a` done in (4.34). Its value is still `25 + 4 y`. The mental model of the state of the session after the operations (4.26) through (4.36) are done is:**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>x</code></td>
<td>4</td>
</tr>
<tr>
<td><code>result2</code></td>
<td><code>25 + 4 y</code></td>
</tr>
</tbody>
</table>
4.8 Troubleshooting assignments

Equations are not the same as assignment

Assignment is an operation that many programming languages have. In some languages (e.g. Maple, Pascal, Eiffel) := is used for the assignment operation. In others (C, Java, Matlab) = is used as the symbol for assignment. Maple uses := because it uses = for equations. It would be confusing to computers and to human readers to use the same symbol for two common but different operations in a single language.

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a := 3 )</td>
<td>We assign ( a ) the value 3.</td>
</tr>
<tr>
<td>( x = 4 )</td>
<td>This is an equation. It doesn't assign ( x ) any value.</td>
</tr>
<tr>
<td>( p := a + x )</td>
<td>We assign ( p ) the value of the expression ( a + x ). ( a ) stands for the value 3 at this point since we did an assignment to it. ( x ) is just a symbol that has no assigned value.</td>
</tr>
<tr>
<td>( x := 47 )</td>
<td>We can do an assignment to ( x ).</td>
</tr>
<tr>
<td>( p := a + x )</td>
<td>This time ( p )'s value is ( 3 + 47 = 50 ).</td>
</tr>
</tbody>
</table>

The name to be assigned always goes on the left hand side of the :=

Since \( 5 = x \) and \( x = 5 \) mean the same thing as mathematical equations, some people think that this should mean that \( x := 5 \) and \( 5 := x \) should both assign the value 5 to \( x \). However, only \( x := 5 \) does the assignment.

<table>
<thead>
<tr>
<th>Assignment := is not symmetric. It matters which side the name is on</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5 := x )</td>
<td>This doesn't mean anything to Maple. The name is supposed to be on the left hand side.</td>
</tr>
<tr>
<td>Error, illegal use of an object as a name</td>
<td></td>
</tr>
<tr>
<td>( x := 5 )</td>
<td>This assigns ( x ) the value 5.</td>
</tr>
<tr>
<td>5</td>
<td>(4.43)</td>
</tr>
<tr>
<td>( z := y )</td>
<td>This assigns ( z ) the (symbolic) value ( y ). It doesn't assign ( y ) any value.</td>
</tr>
<tr>
<td>( y )</td>
<td>(4.44)</td>
</tr>
<tr>
<td>( z + (z + 1)^2 )</td>
<td>(4.45)</td>
</tr>
</tbody>
</table>
To undo all assignments, use restart

Sometimes you want Maple to forget all the assignments you have made in a session. You can get this to happen either by using unassign on each assigned name, or by entering restart in Math mode and then hitting enter. This will unassign everything, undoing all the assignments.

restart does not erase the worksheet, however. The worksheet still looks the same, including the written record of the assignments you had previously done. What the restart does is to delete all the slots you have set up in your mental model.

A subsequent assignment to the same name/variable undoes the previous assignment

Every name/variable can be assigned at most one value at a time. It is permissible to assign a name several times during a sequence of operations, but each assignment replaces the previous association. While the document will record each assignment, only the most recently performed assignment will be in effect if you use a name after it is assigned.

Table 4.7: Reassignment undoes the effect of previous assignment

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a := 3$</td>
<td>This example perform a series of assignments. Its purpose is to demonstrate the effect of reassignment and unassignment. First, we assign $a$ the value 3.</td>
</tr>
<tr>
<td></td>
<td>(4.47)</td>
</tr>
<tr>
<td>$x := 4 + a$</td>
<td>We assign $x$ the value that's 4 plus the value of $a$.</td>
</tr>
<tr>
<td></td>
<td>(4.48)</td>
</tr>
<tr>
<td>$a := 5$</td>
<td>We assign $a$ the value of the expression $a + x$. $a$ stands for the value 3 at this point since we did an assignment to it. $x$ is just a symbol that has no assigned value.</td>
</tr>
<tr>
<td></td>
<td>(4.49)</td>
</tr>
<tr>
<td>$x$</td>
<td>If we ask for the value of $x$ we still get 7, which is what it was set to after the last assignment of $x$ at (1.8.7). Maple does not go back and come up with a new value of $x$ just because $a$ has been assigned subsequent to (1.8.7).</td>
</tr>
<tr>
<td></td>
<td>(4.50)</td>
</tr>
<tr>
<td>$y := x + a + 1 + z$</td>
<td>We can do an assignment to $y$. The expression $x + a + 1$ evaluates to the currently assigned value of $x$ which is 7, plus the most recently assigned value of $a$ which is 5, plus 1, plus the most recently assigned value of $z$. Since $z$ has no assigned value, it is treated as an algebra symbol and left as the symbol $z$.</td>
</tr>
<tr>
<td></td>
<td>(4.51)</td>
</tr>
<tr>
<td>unassign('a')</td>
<td>If we unassign $a$, the previously assigned value is forgotten. But that does not cause the previously assigned values of $y$ and $x$ to be forgotten.</td>
</tr>
<tr>
<td>$a$</td>
<td>(4.52)</td>
</tr>
<tr>
<td>$y + 47$</td>
<td>(4.53)</td>
</tr>
<tr>
<td>$x$</td>
<td>(4.54)</td>
</tr>
</tbody>
</table>
Some names are already used by Maple. You will get an error message if you try to assign to them yourself.

When you first start up Maple, the names that you would ordinarily think of using to assign to are not assigned. However a few are, such as the symbolic constants $\pi$ and $i$. Maple will tell you that such names are reserved for system use. You need to pick another name.

Table 4.8: Maple won't let you use some names it is already using

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi := 47$</td>
<td>You can't redefine a symbolic constant.</td>
</tr>
<tr>
<td>Error, attempting to assign to <code>Pi</code> which is protected</td>
<td></td>
</tr>
<tr>
<td>$\text{for} := 3.1$</td>
<td>$\text{for}$ isn't a symbolic constant but it is used in Maple's programming language. So we can't use it as a variable. We can tell that something funny is going on because the for is automatically turned into a bold for, and a red box appears around the initial part of what we typed.</td>
</tr>
<tr>
<td>$\text{solve} := x + 1$</td>
<td>$\text{solve}$ is the name of operation that solves equations. You can't change its meaning by using it as a variable to assign to.</td>
</tr>
</tbody>
</table>

4.9 Summary of Chapter 4 material

<table>
<thead>
<tr>
<th>Assignment</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>General form</td>
<td>Examples</td>
</tr>
<tr>
<td>Assignment is performed by using the assign to a name operation of the clickable menu.</td>
<td>$x := 5$</td>
</tr>
<tr>
<td></td>
<td>$5$</td>
</tr>
<tr>
<td></td>
<td>$\frac{x^2}{2}$</td>
</tr>
<tr>
<td></td>
<td>$\frac{25}{2}$</td>
</tr>
<tr>
<td>Assignment also be performed by typing in the name, followed by :=, followed by the expression whose value will be the result to be assigned. $\text{symbol name} := \text{expression}$</td>
<td>$x + 1$</td>
</tr>
<tr>
<td></td>
<td>$6$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Unassignment</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>General form</td>
<td>Examples</td>
</tr>
<tr>
<td>unassign(' symbol name ')</td>
<td>$x + 1$</td>
</tr>
<tr>
<td></td>
<td>$6$</td>
</tr>
<tr>
<td>unassign('x')</td>
<td>$x + 1$</td>
</tr>
<tr>
<td></td>
<td>$x + 1$</td>
</tr>
<tr>
<td>restart</td>
<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>Undoes all assignments made by the user in the session so far.</td>
<td></td>
</tr>
</tbody>
</table>
5.1 Chapter Overview

We briefly discuss a few extra concepts useful with \textit{solve}: how to use a combination of relations rather than just a single equation, and how to take apart or combine by the various forms of \textit{solve}.

We then explore the concept of \textit{script}: a sequence of operations useful for solving a problem. We find that it's often the case that the need for a computation is driven by its reuse -- doing the same thing but with slight alterations each time. A frequently recurring scenario is a parameterized computation: 1) use variables to assign values to the parameters and 2) have subsequent steps of the computation refer to the parametric variables. Maple is well-equipped for reuse of parameterized scripts, since it has an operation \textbf{Edit} $\rightarrow$ \textbf{Execute} $\rightarrow$ \textbf{Selection} or \textbf{Worksheet}. This makes it easy to solve different versions of a problem by editing the parameter values and re-executing the script.

5.2 The structure of information in Maple: getting information from solve

The result of the \textit{solve} operation can have multiple parts if there are multiple solutions to the equation. In this case, the result of \textit{solve} is a sequence, list, or set of solutions, and we can select each part by giving an \textit{index} (either 1 or 2).

\[
eg \text{eq1s := 3} \cdot x - x^2 - 28 \quad \text{solve} \quad \{x = -4\}, \{x = 7\} \quad \text{select entry 1} \quad \{x = -4\} \]

\[
eg \text{eq1s := 3} \cdot x - x^2 - 28 \quad \text{solve} \quad \{x = -4\}, \{x = 7\} \quad \text{select entry 2} \quad \{x = 7\} \]

If we give solve a linear equation, it has only one solution. We can still select the first entry.

\[
eg \text{eq2s := 3} \cdot x = 28 \quad \text{solve} \quad \{x = \frac{28}{3}\} \quad \text{select entry 1} \quad x = \frac{28}{3} \]

If we do "solve for x" for the same equation, we see that the answer comes back in a slightly different form. But it still has parts.

\[
eg \text{eq1L = 3} \cdot x - x^2 - 28 \quad \text{solve for x} \quad \{[x = -4], [x = 7]\} \quad \text{select entry 2} \quad [x = 7] \]

Maple (as well as many other programming languages) can compute with objects that have \textit{structure}. Here are four different kinds of structures that Maple can handle:

<table>
<thead>
<tr>
<th>Type of structure</th>
<th>What they look like</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equations, inequalities</td>
<td>What math books do for equations and inequalities. The clickable menu allows you to get at the \textbf{left hand side} and \textbf{right hand side} of an equation or inequality.</td>
<td>[ x + y = 35 ] \hspace{1cm} \text{left hand side} \hspace{1cm} [ x + y ] \hspace{1cm} [ \sqrt{57} \leq x ] \hspace{1cm} \text{right hand side} \hspace{1cm} [ x ]</td>
</tr>
<tr>
<td>Type of structure</td>
<td>What they look like</td>
<td>Examples</td>
</tr>
<tr>
<td>-------------------</td>
<td>--------------------</td>
<td>----------</td>
</tr>
</tbody>
</table>
| Sequences         | Values or expressions separated by a comma | \( x := 19, 47, 92 \)  
|                   |                    | \( 19, 47, 92 \) \text{(5.7)}  
|                   |                    | \( \text{relations} := x^2 - 2 = 0, 0 < x \)  
|                   |                    | \( x^2 - 2 = 0, 0 < x \) \text{(5.8)}  
|                   |                    | select entry 1  
|                   |                    | \( x^2 - 2 = 0 \) \text{(5.9)}  
| Sets              | A sequence surrounded by curly braces \{ \} | \( \text{Scores} := \{ 3, 7, 3, 10 \} \)  
|                   |                    | \( \{3, 7, 10\} \) \text{(5.10)}  
|                   |                    | select entry 3  
|                   |                    | 10 \text{(5.11)}  
| Lists             | A sequence surrounded by square brackets \[ \] | \( \text{MyEquations} := [x + y = 3, y - 2 \cdot x = 37] \)  
|                   |                    | \( [x + y = 3, y - 2x = 37] \) \text{(5.12)}  
|                   |                    | \( \text{solve} \)  
|                   |                    | \( \{ x = -\frac{34}{3}, y = \frac{43}{3} \} \) \text{(5.13)}  

Note that the result of this solve is a set of equations.

For the time being, we just want you to recognize the different kinds of structures that are output by solve and other functions and be able to select parts from them. Later on we will get a lot of work done by performing more sophisticated operations with them.

### 5.3 Finding simultaneous solutions, constraining solutions.

Suppose we want to solve the system of equations \( x + y = 5 \) and \( -3 \cdot y + 7 = x \). This means finding values of \( x \) and \( y \) that simultaneous satisfy both equations. We can do this in Maple by typing in the first equation and then the second, separated by a comma. This is called entering a sequence of equations. Right-clicking (control-click on Macintosh) on the sequence will allow you to solve the system.

In Lab 1, you discovered that solve could also handle inequalities as well as equalities. You can enter a sequence of equations and inequalities to solve. This can be used to limit solutions to a particular range of values.

**Table 5.2: Solving simultaneous equations**

<table>
<thead>
<tr>
<th>( x + y = 5, -3 \cdot y + 7 = x )</th>
<th>The result of this solve is a set of solutions.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + y = 5, -3y + 7 = x )</td>
<td>( {x = 4, y = 1} ) \text{(5.14)}</td>
</tr>
<tr>
<td>( \text{solve} )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( {x = 4, y = 1} ) \text{(5.15)}</td>
</tr>
</tbody>
</table>
\[ p := x^4 + 3x^2 - 57.5 = 0 \]

\[ \text{solve} \]

\[ (x = 3.0380606341), (x = -3.0380606341), (x = 2.495959218), (x = -2.495959218) \]  

Solving this equation produces 4 roots. Two of them are complex numbers (since they have \( i \) in them) the others are real.

\[ x^4 + 3x^2 - 57.5 = 0 \]  

\[ (x = x) \]

\[ \text{solve} \]

\[ x = 2.495959218 \]  

This gets the solver to list only the positive real solutions. We might be interested in only non-negative real solutions, if, for example, we were using the equation to model a physical situation and \( x \) was measuring weight. In this case the other values of \( x \) wouldn't be relevant to our situation.

\[ p, x \geq 0 \]

\[ \text{solve} \]

\[ x^4 + 3x^2 - 57.5 = 0, \; 0 \leq x \]  

\[ (x = 2.495959218) \]

\[ x = 2.495959218 \]

\section*{5.4 Scripting: creating computational work in reusable form}

Consider the problem you did in Lab 1, along with a solution:

\textbf{Version 1 and solution}


A breeding group of 20 bighorn sheep is released in a protected area in Colorado. It is expected that with careful management the number of sheep, \( N \), after \( t \) years will be given by the formula:

\[ N = \frac{220}{1 + 10 \cdot (0.83)^t} \]

and that the sheep population will be able to maintain itself without further supervision once the population reaches a size of 80.

(a) Graph \( N \) versus \( t \).

(b) How many years must the state of Colorado maintain a program to care for the sheep? Show your work.

(c) How many bighorn sheep can the environment in the protected area support? (hint: examine the graph for large values of \( t \).)

\textbf{Solution to (a)}

\[ N = \frac{220}{1 + 10 \cdot (0.83)^t} \quad N = \frac{220}{1 + 10 \cdot 0.83^t} \quad \text{right hand side} \quad \frac{220}{1 + 10 \cdot 0.83^t} \quad \text{assign to a name} \quad \textit{sheepPopExpr} \]
Once we have the sheep population, we need to play with the plotting ranges to see when the leveling off occurs. We’d have to think about it and experiment a bit -- but the computer makes the replotting easy to do once we make our decisions about what to try.

\[ \text{sheepPopExpr} \]

\[
\frac{220}{1 + 10^{0.83t}} \tag{5.22}
\]

\[
# \text{ of sheep versus time}
\]

\[
N \quad 0 \quad 10 \quad 20 \quad 30 \quad 40 \quad 50
\]

\[
t \quad 0 \quad 10 \quad 20 \quad 30 \quad 40 \quad 50
\]

\[
20 \quad 40 \quad 60 \quad 80 \quad 100 \quad 120 \quad 140 \quad 160 \quad 180 \quad 200
\]

**Solution to (b)**

\[ 80 = \text{sheepPopExpr} \]

\[
80 = \frac{220}{1 + 10^{0.83t}} \tag{5.23}
\]

\[
solve \rightarrow \tag{5.24}
\]

\[ \{ t = 9.354227718 \} \]

We can read the leveling off point from the plot, assuming that we have figured out the appropriate time range in (b). Alternatively, we can do a little calculus and take the limit of the expression as \( t \) goes to infinity.

\[
\lim_{t \to \infty} \text{sheepPopExpr} \]

\[ 220. \tag{5.25} \]
We can imagine ourselves working as a environmental engineer for the National Forest Service and being very pleased with ourselves for solving the problem with Maple. But now we are handed two more wildlife management problems to do, from other regions in our territory:

Version 2

A breeding group of 33 bighorn sheep is released in a protected area in Colorado. It is expected that with careful management the number of sheep, \( N \), after \( t \) years will be given by the formula:

\[
N = \frac{330}{1 + 10 \cdot (0.79)^t}
\]

and that the sheep population will be able to maintain itself without further supervision once the population reaches a size of 85.

(a) Graph \( N \) versus \( t \).
(b) How many years must the state of Colorado maintain a program to care for the sheep? Show your work.
(c) How many bighorn sheep can the environment in the protected area support? (Hint: examine the graph for large values of \( t \)).

Version 3

A breeding group of 45 bighorn sheep is released in a protected area in Colorado. It is expected that with careful management the number of sheep, \( N \), after \( t \) years will be given by the formula:

\[
N = \frac{450}{1 + 10 \cdot (0.63)^t}
\]

and that the sheep population will be able to maintain itself without further supervision once the population reaches a size of 90.

(a) Graph \( N \) versus \( t \).
(b) How many years must the state of Montana maintain a program to care for the sheep? Show your work.
(c) How many bighorn sheep can the environment in the protected area support? (Hint: examine the graph for large values of \( t \)).

We have the feeling that we will shortly be handed problems for a number of other locations as well. How can we reuse our original work with minimal effort?

If we had the first solution, we could produce the second solution through *copy-paste-edit-re-execute*:

<table>
<thead>
<tr>
<th>Executing a clone of a script through copy-paste</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Copy and paste the original solution into a new document, or to a spot later in the same document. You do this as with a word processor -- select a region of the worksheet with the mouse, type control-C (command-C on a Mac). Then position the document cursor in the destination, and type control-V (command-V on a Mac). A duplicate of the solution should appear there.</td>
</tr>
<tr>
<td>2. Using the mouse, edit the formulae as needed. You only need to edit the black (input) formulae, not the blue ones.</td>
</tr>
<tr>
<td>3. Position the mouse at the first computation and hit <em>enter</em>. Continue to work your way through the sequence of the commands.</td>
</tr>
<tr>
<td>4. Alternatively, select the entire region containing the edited version of the solution and hit <em>Edit-&gt;Execute-&gt;Selection</em>.</td>
</tr>
<tr>
<td>5. If the region to be executed is the entire worksheet, then rather than selecting anything you can do <em>Edit-&gt;Execute-&gt;Worksheet</em>.</td>
</tr>
</tbody>
</table>

The results of executing the edited script are not totally correct. We will have to change a few things by hand: the clickable operation "evaluate at a point" in step (b) will use \( N=80 \) (which is what the copy says) instead of \( N=85 \), so we will have to redo it. Also, the last plot will not show an appropriate vertical range unless we do that. We'd like to do a little less of this hand-tuning in re-execution. Also, we'd like to make it easier to remember what to change between multiple versions.

*A breeding group* (page 58).
5.5 Rewriting the script using assignment

While copying and editing is probably a little faster than typing in the whole script again, we can reduce the amount of hunting around for changes by writing a script finding the parameters of the problem, and writing the script so that it assigns values to the parameters at the beginning of the problem.

You may be doing assignments at several points in your calculation. Only the ones that would need to be changed between different versions of the problem define problem parameters. You may find the other ones very useful, but they don't have parameter status.

Finding and naming parameters

First, solve at least one version of the problem. Then, imagine what would need to be changed if you were trying to solve alternative versions of the problem. You can find parameters if you have several versions of a problem by looking at what changes in the worksheet from version to version.

For example, in the sheep problem, we note the following things changing in different versions of the problem. We pick names for these.

1. the numerator of the "sheep equation" \( P \)
2. the coefficient in the denominator of the equation \( c \)
3. the value of the stable population \( s \)

The other number in the script is the original population of the sheep. We realize that it can be derived by evaluating the "sheep equation" at \( t=0 \). Thus this value does not need to be a parameter as we can derive it from the other information. It might be a good check though.

We then write the script to assign values to the parameters at the start of the script, and then write the other operations and expressions in terms of the symbols.

This allows us to redo the script just by changing the values at the beginning of the script. We use the word processing features to add extra directions to make the script easier to use, basically saying "here, change these things", and "this is the end of where you should stop copying".

The result of executing the script is Version 2, with use of parameters (page 59).

Having created this script, we can handle the third version of the problem by editing the values of the parameters and executing the script again by selecting the entire script with the mouse, and then doing Edit->Execute->Selection. We get this result:

5.6 Summary of script writing

Script writing is appropriate when you expect to handle several different versions of the same problem. In professional work, this is often the case -- if it's worth doing at all, it's probably going to occur more than once.

Figure out how to solve the problem first. Then write the script. There's really not much point in writing the script if you don't have some idea of the sequence of operations in it.

Once you have a worksheet of instructions for solving one version of the problem, look at it and the other versions and find the parameters. Set up a new worksheet where the first thing you do is to assign values to the parameters. Then work through the rest of the instructions and rewrite them to use the parameters instead of the fixed values from one version of the problem.
5.7 Troubleshooting scripts

Programming books have the tendency to show things that work. They say less about what to do about the things that don't work. However, it is usually the case that computer users often spend more time "getting things to work" than "working". Learning how to get out of jams is at least as valuable as knowing how to enter operations and what they mean. We offer this advice, which is "commonsense about work" as applied to interactive software development:

1. Solve one version of the problem before you try to start scripting. You can use Maple to experiment -- enter and edit snippets of operations that try out the solution technique for part of the problem. Eventually edit them together so that they solve the whole problem. If you have only fuzzy notions about the math or the operations you want to proceed, your computer work will just amplify that. Having a worksheet that solves one version of the problem can remove a lot of the fuzziness.

Where does the inspiration for solving the problem come from? If you are lucky, the solution may be told to you. Or you may find a description of a similar problem as a starter. But the big bucks, as they say, go to those who can devise the solution plan themselves.

2. Limit each step so that it is a small step. If you get into trouble, you will be able to nip it at the bud. This becomes a more prominent tactic in later work when we are tempted to construct long-winded one-line expressions that do everything at once.

3. Test pieces individually, then put them together. For example, if you don't really understand how to make a dotted line plot in teal with a title, you should try to do that with a simple plot (such as $x^2$) rather than an expression whose shape you aren't that familiar with it. Then take what works and substitute the real expression you want to plot in a copy of the $x^2$ plot operation.

If you think about it, this is similar to what happened in Fall 2010 ENGR 101 Lab 2, where they first had you learn how to trigger an oscilloscope with the output of the function generator, rather than the output of the camera/flash sensor. The complexity of troubleshooting is reduced if you half as many unknowns to worry about. This can be called divide and conquer troubleshooting.

4. If what you have doesn't work, find something similar which does work, and then incrementally edit it. For example, if you can't get $a := x^2 + 3 \cdot x + 1$ in, then first see whether you can get $a := 1$ to work. Once you succeed with that, edit the expression to $a := 3 \cdot x + 1$ and so forth.

5. If strange things continue to happen despite your best efforts to troubleshoot, it may be that previous settings in the Maple session are interfering with your current work. Recall that some people in the ENGR101 lab couldn't get their oscilloscopes to work because of settings changed in the oscilloscopes by groups earlier in the day. This can be particularly true if you are developing scripts and are assigning parameter values, then switch to development of another script in the same document. The values you assigned will not magically unassign themselves when you start working on something new in the same worksheet.

The remedy for this is to put a restart in as the first operation in your script, then re-execute the worksheet.
5.8 Attachments

Attachment: Version 2 of sheep script without parameters

Version 2 of sheep problem, with edited script

A breeding group of 30 bighorn sheep is released in a protected area in Colorado. It is expected that with careful management the number of sheep, \( N \), after \( t \) years will be given by the formula:

\[
N = \frac{330}{1 + 10 \cdot (0.79)^t}
\]

and that the sheep population will be able to maintain itself without further supervision once the population reaches a size of 85.

(a) Graph \( N \) versus \( t \).
(b) How many years must the state of Colorado maintain a program to care for the sheep? Show your work.
(c) How many bighorn sheep can the environment in the protected area support? (hint: examine the graph for large values of \( t \).

Solution to (a)

\[
N = \frac{330}{1 + 10 \cdot (0.79)^t} \quad \text{right hand side} \quad \frac{330}{1 + 10 \cdot 0.79^t} \quad \text{assign to a name} \quad \text{sheepPopExpr}
\]

\[
\text{sheepPopExpr} \rightarrow \frac{330}{1 + 10 \cdot 0.79^t}
\]

Solution to (b)
85 = \textit{sheepPopExpr}

\begin{align*}
85 &= \frac{330}{1 + 10^{0.79^t}} \\
\text{solve} \\
\{ t = 5.277302835 \} \\
\end{align*} \tag{5.28}

We can read the leveling off point from the plot, assuming that we have figured out the appropriate time range in (b). Alternatively, we can do a little calculus and take the limit of the expression as \( t \) goes to infinity.

\begin{align*}
\lim_{t \to \infty} \textit{sheepPopExpr} \\
&= 330. \\
\end{align*} \tag{5.30}

**Attachment: Version 2 of Sheep Script, with parameters**

**Version 2, with use of parameters**

**Start of parameters -- change these for each version of the problem**

\( P := 330 \)

\begin{align*}
330 \\
\end{align*} \tag{5.31}

\( c := 0.79 \)

\begin{align*}
0.79 \\
\end{align*} \tag{5.32}

We call the size of the stable population \( s \).

\( s := 85 \)

\begin{align*}
85 \\
\end{align*} \tag{5.33}

**End of parameters**

(a) Graph \( N \) versus \( t \).

(b) How many years must the state of Colorado maintain a program to care for the sheep? Show your work.

(c) How many bighorn sheep can the environment in the protected area support? (hint: examine the graph for large values of \( t \).)

(a)

\begin{align*}
\textit{sheepPopExpr} &:= \frac{P}{1 + 10^{c^t}} \\
&= \frac{450}{1 + 10^{0.83^t}} \\
\end{align*} \tag{5.34}
Note that this `sheepPopExpr` is not a parameter since assignment is always the same for all versions of the problem.

To make the graphing work all the time, we set the vertical axis to "P+30" rather than a fixed value. This will set the vertical axis so that it will be 30 more than the value of P which from the first problem we have realized is the top of the graph.

\[ \frac{330}{1 + 10^{0.79t}} \]  
\( \rightarrow \)

\[ s = sheepPopExpr \]

\[ 85 = \frac{330}{1 + 10^{0.79t}} \]

\( \text{solve} \)

\( \{ t = 5.277302835 \} \)

This is the time in years that model predicts it will take for the sheep population to reach self-sustaining status, allowing the wildlife managers to move onto another job.

(c) A little thinking reveals that the leveling off value (as indicated from the graph), is the numerator of `sheepPopExpr`, since the denominator goes to 1 as \( t \) goes to infinity.

\[ \lim_{t \to \infty} sheepPopExpr \]

\[ 330. \]
End of script

**Attachment: Version 3 of Sheep Script, with parameters**

Version 3 with edited parameters and re-execution

Start of parameters -- change these for each version of the problem

\[ P := 450 \]

\[ c := 0.83 \]

\[ \text{sheepEquation} := N = \frac{P}{1 + 10 \cdot (c)^f} \]

\[ N = \frac{450}{1 + 10 \cdot 0.83^f} \]

We call the size of the stable population \( s \).

\[ s := 100 \]

End of parameters

(a) Graph \( N \) versus \( t \).

(b) How many years must the state of Colorado maintain a program to care for the sheep? Show your work.

(c) How many bighorn sheep can the environment in the protected area support? (hint: examine the graph for large values of \( t \).)

\[ \text{sheepPopExpr} := \frac{P}{1 + 10 \cdot (c)^f} \]

Note that this sheepPopExpr is not a parameter since assignment is always the same for all versions of the problem.

To make the graphing work all the time, we set the vertical axis to "P+30" rather than a fixed value. This will set the vertical axis so that it will be 30 more than the value of \( P \) which from the first problem we have realized is the top of the graph.

\[ \text{sheepPopExpr} = \frac{450}{1 + 10 \cdot 0.83^f} \]
This is the time in years that model predicts it will take for the sheep population to reach self-sustaining status, allowing the wildlife managers to move onto another job.

(c) A little thinking reveals that the leveling off value (as indicated from the graph), is the numerator of $\text{sheepPopExpr}$, since the denominator goes to 1 as $t$ goes to infinity.

$$\lim_{t \to \infty} \text{sheepPopExpr} = 450.$$  \hfill (5.45)
5.9 Summary of Chapter 5 material

<table>
<thead>
<tr>
<th>Type of structure</th>
<th>What they look like</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic data structures in Maple</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| Equations and inequalities | Expression related by $=$, $>$, $<$, $\geq$, $\leq$, or $\neq$. | $x + y = 0$
  $n^2 - 3n > 4$
  $4 < n^2 - 3n$ (5.46) |
| Sequences                  | Values separated by a comma | 19, 47, 92
  19, 47, 92 (5.48) |
| Lists                      | A sequence surrounded by square brackets $[\ ]$ | $MyData := [1.0, x, \frac{3}{4}, a]$
  $[1.0, x, \frac{3}{4}, a]$ (5.49) |
| Sets                       | A sequence surrounded by curly braces $\{\}$ | $Scores := \{3, 7, 3, 10\}$
  $(3, 7, 10)$ (5.50) |

Solving simultaneous equations

$x + y = 5, -3y + 7 = x$

\[ x + y = 5, -3y + 7 = x \] (5.51)

\[ \text{solve} \]

\[ \{x = 4, y = 1\} \] (5.52)

$p := x^4 + 3x^2 - 57.5 = 0$

\[ x^4 + 3x^2 - 57.5 = 0 \] (5.53)

\[ \text{solve} \]

\[ \{x = 3.0380606341, x = -3.0380606341, x = 2.495959218, x = -2.495959218\} \] (5.54)

$p, x \geq 0$

\[ x^4 + 3x^2 - 57.5 = 0, 0 \leq x \] (5.55)

\[ \text{solve} \]

\[ \{x = 2.495959218\} \] (5.56)

The result of this \textit{solve} is a set of solutions.

Solving this equation produces 4 roots. Two of them are complex numbers (since they have $i$ in them) the others are real.

This gets the solver to list only the positive real solutions. We might be interested in only non-negative real solutions, if, for example, we were using the equation to model a physical situation and $x$ was measuring weight. In this case the other values of $x$ wouldn't be relevant to our situation.
In a Maple worksheet, take a version of a problem and create a sequence of operations in the worksheet that solve it.

Note similarities and differences between different versions of the problem. Envision what you'd have to change in the worksheet in order to solve a different version of the problem, and what would stay the same. You may have to rewrite some of the expressions to refer to the parameter rather than the value.

Assign the parameters at the beginning of the script. Rework the rest of the script so that the formulas refer to the parameters by name, rather than the values used in the original version of the problem.

For example, if the value 42 appears in several places in your script, define a parameter \( p := 42 \) at the start of the script and edit the other occurrences of 42 to be \( p \) instead. When you have a different version of the problem, you can edit just the single line \( p := 42 \) into say \( p := 47 \) and won't need to edit any other lines of the script.

<table>
<thead>
<tr>
<th>Rationale for using scripts</th>
<th>More work to do than clickable interface the first time. Saves time if you expect to want to reuse the operations on multiple versions of the same problem. Also it is less error prone.</th>
</tr>
</thead>
</table>
6 Chapter 6 More sophisticated scripting

6.1 Chapter Overview

We introduce textual entry of solve and plot operations. This is where the operation is specified by keyboard entry alone, without the use of the mouse or Palettes. Textual entry is often preferred by programmers because it is easier to edit scripts written through textual means. We retain the clickable interface for doing quick one-time calculations, or for developing ideas the first time before we start script-writing.

We begin to introduce additional concepts in Maple, to enhance what we can solve and plot:

1. In Maple, a character string is a collection of characters delimited by "s: "This is a string." We see how strings are used in the textual entry of labels and colors in plots.

2. Lists e.g. [1,2, x, 3.5] provide a way of organizing multiple results in a single "data container", making it to operate on the whole collection of results while retaining the ability to getting at individual results from within the collection. Lists are used in both solve and plot.

3. solve uses two other types of data containers: sequences and sets. We describe how to recognize them, and how to extract information from them.

Programmers rely on the on-line documentation to manage the complexity of remembering the details of a knowledge-intensive system such as Maple. They learn/remember how to use a feature by looking up the description, finding an example close to what is desired, and then actively experiment with the example in a fresh worksheet. Reading without experimentation is usually not very productive.

In a previous chapter, we explained assignment and how Maple uses assigned values whenever it sees a name in an expression. We introduce the eval operation, which allows assignments to be done temporarily and immediately forgotten. This can be an attractive alternative if you are concerned about situations where you'd be making many assignments and then undoing them through unassign. eval allows you to evaluate an expression for a particular value of a variable in one line, rather than having to type in the assignment and unassignment as well.

6.2 Textual entry of operations

The textual form of an operation in Maple has the general form:

operationName( sequence of values )

The operationName can be something like solve or plot. By sequence of values, we mean one or more items, each item separated from the next by a comma. "Sequence" here is the same kind of sequence that was first seen in the previous section on solve (page 52).

Maple will evaluate what you enter in the same way that was described for mathematical expressions possible (page 35). If there are assigned variables mentioned in the sequence, then their values will be used. If Maple knows the operationName (e.g. sin, solve, plot), then it will perform the calculation specified by the built-in programming. Otherwise, the result will be more or less what you typed in.

The technical term for the "values" in this situation is actual parameter or argument. Note that the parentheses around the sequence of values are mandatory -- you will either get an error or a result that's far from what you want if you omit the parentheses.

This style of writing things is sometimes called functional notation. In mathematics examples of functional notation are f(x) or g(3,5). In these examples, the name of the operation is the function name f or g, while the actual parameters or arguments would be x or the sequence 3,5.

Example of textual form of equation solver solve.
The form of the answer returned by the textual version is slightly different from invoking solve through the clickable interface. The former is typically a form that is easier to work with in scripts.

In this, the operation Name (or function name) is `solve`. There are two arguments. The first argument is the equation \( x = 3 \cdot x^2 - 2 \). The second argument is the symbol \( x \).

### Examples of the textual version of plotting

This is a textual form of plot. The first argument is an expression, the second argument is an equation naming a variable and a range of a plot.

Note that if we wanted to change the range from -3..3 to -5..2 then we would just edit that line of the worksheet and hit enter again. If we wanted to redo the plot in the clickable interface, we would have to right-click and enter all the information all over again.
If the second argument is just the variable, then `plot` uses default values for the range.

```
plot(x - 3\cdot x^2 - 2, x)
```

No plotting happens if we forget the mandatory parentheses.

```
plot x - 3\cdot x^2 - 2, x = -3 \cdot 3
```

Suppose we entered this instead of `plot(x - 3\cdot x^2 - 2, x = -3 \cdot 3)` . There is no error message, but the picture is not at all the same.

This example illustrates the fact that just because there is no error message, it does not mean that the computer will produce what you thought you were asking for. The only way that we would discover the mistake is if we already had an idea of what the graph should look like, and noticed that the result differed significantly from what we expected. If you haven’t formed a basis for expectations, you won’t discover the problem.

Once you realize that the picture must be incorrect, you would be spurred to search for the cause. Since there are no error messages and a plot (albeit a weird one) was produced, the most likely cause is that the arguments to the plot function are wrong. The obvious place to look for correct examples is the on-line documentation. If you look at the on-line documentation for help, you will see that giving `plot` three arguments means something different -- the third argument can be taken as the value of the vertical range of a plot. Evidently what is happening is that you are seeing only the tiniest top slice of the plot produced above because of the inadvertent specification of the vertical range.

An attachment at the end of the chapter shows the textual form of common functions, subscripts. These textual forms can be entered from the keyboard wherever the palette entry would work.
6.3 Why are there two different styles for entering operations?

The clickable interface is a good way to get a calculation done quickly, but the actions specified in this way are hard to edit when building scripts. Maple, like most languages, has a textual version of all operations it performs. The editing involved in scripting development can often be easier to do on the textual version. In other words, the clickable interface is good for a one-time calculation, but not so good for the editing and re-execution involved in script reuse.

Another advantage of the textual mode of operation is that the number and variety of operations available in textual form is far greater than what's available in the clickable interface. Maple has several thousand operations. Building a clickable interface to all of them would result in tedious navigation through menus that would either be huge or involve many sublevels.

The downside of using the textual entry is that the developer must spell the text correctly, with the right number and placement of parentheses. Experienced users find that the textual interface is faster to deal with for scripting, while the clickable interface is faster for short, more casual use. Fortunately, in either case one can edit failed attempts and retry, so perfect entry is not necessary to be productive.

Becoming proficient with textual entry of operations is part of the transition technical users make in going from just reuse of other's work to routinely creating their own programming. Without such proficiency, it is hard to realize the full power of the computer in modeling and simulation situations.

6.4 Plotting a list of expressions (multi-plots), plotting lists of numbers (point plots)

Recall that lists in Maple are a way of collecting expressions together into a single object, as discussed in the previous describing lists and other data containers (page 52). You specify a list by listing the items in the list , enclosed in square brackets [ ].

If the first argument to plot is a list of expressions, then plot will on a single graph display the plots of all the expressions in the collection. Each one will be displayed in a different color.
### Table 6.1: Plotting of multiple expressions

<table>
<thead>
<tr>
<th>Example</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>plot([x^2, sin(x^2)], x = 0 .. 4)</code></td>
<td>Plot two expressions on a range where both have comparable-sized results. Here we use the list <code>[x^2, sin(x^2)]</code> to indicate the two expressions that should be plotted.</td>
</tr>
<tr>
<td><code>plot([-5, x - 3*x^2 - 2], x = -3 .. 3)</code></td>
<td>Problem: approximately where is the expression <code>-3*x^2 + x - 2</code> equal to 5? While we could use solve to tell us exactly, it's often worthwhile to draw a picture and process the situation visually. If we give plot the two expressions &quot;-5&quot; and <code>-3*x^2 + x - 2</code> then plot will plot not only the parabola (the second expression, it will also plot the expression that is always -5 for any value of x. This corresponds to the horizontal line drawn on the plot. Visually we can see that the parabola is -5 at roughly -.8 and 1.2. We can even get a little more precise by ???.</td>
</tr>
</tbody>
</table>

Plotting multiple expressions simultaneously can be useful when you want to compare them. Assuming that the scales are comparable, one can get a sense of similarity or dissimilarity "at a glance".
We can plot data points rather than smooth curves, if we give the textual form of plot separate lists of $x$ and $y$ coordinates. can be used with the textual version of `plot`. If we give the textual version of the `plot` operation two lists of numbers that have the same length, then `plot` will regard the first list as a list of $x$-coordinates, and the second list as corresponding $y$-coordinates. If you provide `plot` with the third argument `style=point`, then it will produce a point plot. Otherwise, it will draw lines connecting each point.

**Table 6.2: Plotting points with lists of numbers**

<table>
<thead>
<tr>
<th>xList := [1, 2, 3, 4]</th>
<th>yList := [5, 6, 7, -1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[1, 2, 3, 4]$</td>
<td>$[5, 6, 7, -1]$</td>
</tr>
</tbody>
</table>

We use the textual form of `plot` to plot the points (1,5), (2,6), (3,7) and (4,-1).
Without the third argument, `plot` will try to connect the points with a curve.

Maple cares about whether things are capitalized or not. `Style` is not the same as `style`.

### 6.5 Strings -- a way to specify titles and labels in plots

A string in Maple is something enclosed in double-quotes: "red", "this is a string?", "Blink++++++182+++++" are all strings. The double-quote symbol is mandatory for a string. Single-quotes ’ (also known as apostrophes), backquotes ` (also known as acute accent marks) are not substitutes for double-quotes in writing Maple strings. Characters enclosed by apostrophes or backquotes mean something different to Maple. Use of the wrong punctuation marks will lead to undesired results or error messages.

In addition to being used to input data points in `plot`, lists also can be used in specifying colors and axis labels.
Table 6.3: Plot options and labels

If one of the arguments to the plot operation is of the form 
\[ color = \text{list of color names} \] then those colors will be used. Most reasonable names will work, but the full list can be seen in online help (search for colornames). Note that we are using the textual version of the symbolic constant π. If you can remember how to spell it, it can be easier than selecting it from the Common Symbols Palette.

Note that because the expressions being plotted are given in a set, the color assignment is not in the same order that they were typed in. If we wanted the same order, we should give the plot expressions in a list.

Forgetting brackets for the list of colors.

One of the proficiency issues with textual input is that you have to remember all the ( ) parentheses and [ ] brackets. Can you find the missing delimiter(s)?

One reason why there was no error message about delimiters is that 72 • 6 Chapter 6 More sophisticated scripting

If you leave enough delimiters out, you get error messages that don’t complain about missing delimiters. You have to figure out what the problem is, which might involve a missing parentheses even if the message doesn’t say so.

The "Error, (in sin)" is a cue that you should look at the places where you included sin in your text and inspect it for problems. It doesn’t take too much effort for you to notice that there’s no finishing parentheses in the first sin(x).

One reason why there was no error message about delimiters is that
there are multiple missing parentheses. Because there are equal numbers of missing left and right parentheses, there was no alarm for missing delimiters.

```
plot\left(\sin(x), \sin\left(\frac{x}{2}\right), \sin(2\cdot x) \mid x = -4\cdot \pi \cdot 4\cdot \pi, \\
\text{color} = \text{color} = ["Red", "Green", "Blue"]\right)
```

The problem with this plot is that pi doesn't mean the same thing to Maple as Pi. Maple is case-sensitive. Only Pi means the symbolic math constant having to do with the circumference of a circle.

```
plot(3.5\cdot x^2 - 2, x = 1..5, labels = ["temperature (in degrees C)", 
"pressure in kilopascals")
```

If one of the arguments to the plot operation is of the form 
labels = list of axes titles then those will be used. Each title needs to be a string.

In subsequent work, we will see strings used in other situations within Maple other than for plot titles.

### 6.6 Troubleshooting with strings

The most common mistakes with strings is to leave out the delimiting "s, or to use the wrong kind of delimiters. While the similar-looking keyboard characters ' (single quote or apostrophe), and ` (acute accent or backquote) look like would be equivalent, they are use for other purposes in Maple.

<table>
<thead>
<tr>
<th>What happens when you forget to use the &quot; delimiter in strong, or use the wrong character for the delimiter.</th>
</tr>
</thead>
</table>
| plot(3.5\cdot x^2 - 2, x = 1..5, labels = [temperature ( 
    \text{in degrees C}), "pressure in kilopascals")]) |
| Error, invalid in 
```
plot(3.5\cdot x^2 - 2, x = 1..5, labels = [temperature ( 
    \text{in degrees C}), "pressure in kilopascals")])```
|

Forgetting to include "s around one of the titles gives a cryptic error message about an "invalid in". If you were an experienced Maple user, you'd know that in is part of the Maple programming language, and should never be flagged in a string. This would be a clue that there's something wrong around where you entered the first label.

The message is not very helpful about telling you how to fix the mistake, though. This unfortunately is typical in most computer programming languages, despite several decades' effort in building systems software to help people program.
Putting the wrong kind of quote -- ' instead of " didn't make a string. We got the same indication of a problem as before even though the problem is "wrong kind of quote" rather than "no quote".

6.7 Learning through on-line documentation and experimentation

All the options available in the Plot Builder available through the right-click (control-click) interface are also available in the textual version of plot. In fact, there are many additional options and varieties of plotting available. The way to find out what the features are and how to invoke them is to consult the on-line documentation.

We can find out more about the textual forms of plotting by invoking Help -> Maple Help and typing plot into the search field. When we do so, we see the information in the figure below:

Table 6.4: Plot command help

![Maple 13 Help - plot](image-url)
We scroll to the bottom of the page and find an example of this. We are looking for a version of plots where v1 and v2 are lists. We don't see something exactly like that but we do see something with Vectors which are similar. Since the document says this should work for lists or vectors, we take the example and see if we can modify it for our own purposes:

Table 6.5: Examples of plot

Evidently, the first list is the values of the $x$ (horizontal) coordinates, and the second list the values of the $y$ (vertical) coordinate. We copy and paste the example into a Maple worksheet and then see if we can get it to work.
According to the documentation, we should be able to get this to work if the first two arguments are lists or vectors. So we edit the example to do lists instead and re-execute the line to see if it works in the same way.
To learn about plot options such as colors and labels, we click on the plot, options item under the search results for plot (see green oval in the figure). Clicking on that item produces this information. We see information about color (with another link to see colors), along with possibilities, for labels, symbols, styles, etc. Again, the way to learn the options is through copying and pasting the examples into a fresh worksheet, getting them to work, and then modifying them to suit your own purposes.
6.8 More operations on lists

So far we have talked only about creating lists, and assigning lists as the value of a variable. You can also generate a sublist of a list, find a particular item in a list by its position index, count the number of items in the list, and convert a list into other types of data.

Table 6.9: Operations on lists

<table>
<thead>
<tr>
<th>Operation</th>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Create a list</td>
<td>[a, b, c, a]</td>
<td>Lists can contain symbols, numbers, expressions -- anything, even other lists.</td>
</tr>
<tr>
<td></td>
<td>[a, b, c, a]</td>
<td>(6.12)</td>
</tr>
<tr>
<td></td>
<td>[1, 3.47, 97, -5.9, 2.1]</td>
<td>(6.13)</td>
</tr>
<tr>
<td>Specify a sublist of values</td>
<td>[a, b, c]</td>
<td>If a list is followed by another pair of braces with a range inside, then a sublist is computed as a result. Here we have the list that's the first through third items of s1.</td>
</tr>
<tr>
<td></td>
<td>[a, b, c]</td>
<td>(6.14)</td>
</tr>
</tbody>
</table>
### 6.9 solve, lists and sequences

To solve a system of equations, use a list of expressions or equations for the first argument to `solve`. Use a list of variables as the second argument.

`solve` will return a sequence of lists as the result.

When `solve` finds two solutions for an equation (such as if the equation is quadratic), it will return a sequence of solutions. You can recognize a sequence and distinguish it from a list because, the sequence is missing the enclosing brackets `[ ]` that a list has.

Part-selection operations work in sequences in a similar fashion as they were described here (page 79).

In `solve`, lists and sequences look as if they are almost interchangable. Later on we will see situations where lists and sequences must be handled differently.
Table 6.10: Solving systems of equations with solve

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ eqn := 3 \cdot x = x^2 - 28 ]</td>
<td>We get a sequence of solutions since there is a double root.</td>
</tr>
<tr>
<td>[ solns := solve(eqn, x) ]</td>
<td></td>
</tr>
<tr>
<td>[-4, 7 ]</td>
<td>(6.25)</td>
</tr>
<tr>
<td>[ eval(eqn, x = solns[1]) ]</td>
<td>We evaluate the equation at the first root and see that it does satisfy the equation.</td>
</tr>
<tr>
<td>[-12 = -12 ]</td>
<td>(6.26)</td>
</tr>
<tr>
<td>[ eval(eqn, x = solns[2]) ]</td>
<td>Same for the second.</td>
</tr>
<tr>
<td>[ system := [3 \cdot x + 5 \cdot y = 6, 2 \cdot x - 5 = y] ]</td>
<td>We want to assign a set as the value of the variable system. Maple tells us that the name is already in use as a built-in function, so it won't let us do that.</td>
</tr>
<tr>
<td>[ sys := [3 \cdot x + 5 \cdot y = 6, 2 \cdot x - 5 = y] ]</td>
<td>We choose a different variable to assign the set to.</td>
</tr>
<tr>
<td>[ [3 \cdot x + 5 \cdot y = 6, 2 \cdot x - 5 = y] ]</td>
<td>(6.28)</td>
</tr>
<tr>
<td>[ vars := [x, y] ]</td>
<td>We specify the set of variables we want to solve for, then call solve. We get a list with one element (which itself is a list) as a solution.</td>
</tr>
<tr>
<td>[ [x, y] ]</td>
<td>(6.29)</td>
</tr>
<tr>
<td>[ solns := solve(sys, vars) ]</td>
<td></td>
</tr>
<tr>
<td>[ \left[ \left[ x = \frac{31}{13}, y = -\frac{3}{13} \right] \right] ]</td>
<td>(6.30)</td>
</tr>
<tr>
<td>[ soln1 := solns[1] ]</td>
<td>We extract the first element of the list. Notice that that are fewer [ ]s.</td>
</tr>
<tr>
<td>[ \left[ x = \frac{31}{13}, y = -\frac{3}{13} \right] ]</td>
<td>(6.31)</td>
</tr>
<tr>
<td>[ eval(sys, soln1) ]</td>
<td>We evaluate the system at the solution that solve has found and verify that this really does satisfy the system of equations.</td>
</tr>
<tr>
<td>[ \left[ 6 = 6, -\frac{3}{13} = -\frac{3}{13} \right] ]</td>
<td>(6.32)</td>
</tr>
<tr>
<td>[ sys2 := [x^2 + y^2 = 25, x + y = 5] ]</td>
<td>This system of equations has two distinct solutions, so we get a list with two elements in it. Each element is a distinct solution.</td>
</tr>
<tr>
<td>[ [x^2 + y^2 = 25, x + y = 5] ]</td>
<td>(6.33)</td>
</tr>
<tr>
<td>[ solve(sys2, [x, y]) ]</td>
<td></td>
</tr>
<tr>
<td>[ [[x = 5, y = 0], [x = 0, y = 5]] ]</td>
<td>(6.34)</td>
</tr>
<tr>
<td>[ eqn2 := x^2 - 3 \cdot x = 5 ]</td>
<td>We can find the non-negative roots of an equation by including the appropriate inequality in the list of relations given to solve. The result by default is a set. however, if the second argument to solve is a list of variables, then the result will come back as a list. You can select items from a set using the same notation as with lists.</td>
</tr>
<tr>
<td>[ solve(eqn2) ]</td>
<td></td>
</tr>
<tr>
<td>[ eqn2 := x^2 - 3 \cdot x = 5 ]</td>
<td>(6.35)</td>
</tr>
</tbody>
</table>
6.10 Evaluation, eval, and assignment

Suppose that we had an expression relating time $t$ to voltage registered by a capacitor as it is being charged by a battery. In a mathematics or electrical engineering textbook, we might see this written as

$$V(t) = 35 + (65 - 35) \cdot \left(1 - e^{-\frac{t}{3}}\right).$$

We are interested in taking this expression for voltage and doing several calculations with it -- plotting it for a range of $t$, finding values of $t$ that correspond to a specified voltage (e.g. "find the time $t$ when the voltage reached 55 volts"), or finding a voltage corresponding to a specified time (e.g. "find the voltage at $t=2.5$ minutes after the start").

If we set up an assignment in Maple, then we could calculate the voltage at $t=2.5$ minutes by assigning $t$ the value 2.5 and then evaluating $V$. The second evaluation will cause the current value of $t$ to be used. However, if we wanted to plot the expression $V$ after that, then we'd have problems because whenever we would type $t$, Maple would use the value of $t$ rather than the symbol $t$.

### Evaluating an expression using a particular value of one of the variables in the expression, and then plotting

<table>
<thead>
<tr>
<th>$V := 35 + (65 - 35) \cdot \left(1 - e^{-\frac{t}{3}}\right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V := 65 - 30 e^{-\frac{t}{3}}$</td>
</tr>
<tr>
<td>$t := 2.5$</td>
</tr>
<tr>
<td>$t := 2.5$</td>
</tr>
<tr>
<td>$V$</td>
</tr>
<tr>
<td>51.96205374</td>
</tr>
</tbody>
</table>

$$\frac{3}{2} + \sqrt{\frac{29}{2}} \cdot \frac{3}{2} - \sqrt{\frac{29}{2}}$$

$$sys3 := \{eqn2_{,} x \geq 0\}$$

$$sys3 := \{x^2 - 3x = 5, 0 \leq x\}$$

solve(sys3, x)

$$\left\{x = \frac{3}{2} + \frac{\sqrt{29}}{2}\right\}$$

sohnSet := solve(sys3)

$$sohnSet := \left\{x = \frac{3}{2} + \frac{\sqrt{29}}{2}\right\}$$

sohnSet[1]

$$x = \frac{3}{2} + \frac{\sqrt{29}}{2}$$
Rather than \( t=0\ldots0.10 \) Maple is seeing \( 2.5=0\ldots10 \) because it is using the value of \( t \) when evaluating what we typed.

We can clear the path for plotting by unassigning \( t \) first. Note that if we did a \texttt{restart} instead of an unassign we would lose the assignment to \( V \): restarting at this point is a bad idea, since it forces us to redefine \( V \) as well as \( t \).

The same problem would happen if we tried to solve an equation involving \( V \) if we had already assigned \( t \) a value.

### Evaluating an expression using a particular value of one of the variables in the expression, and then solving

**restart**

\[ V := 35 + (65 - 35) \cdot \left(1 - e^{-\frac{t}{3}}\right) \]

\[ t := 4.7 \]

\[ V \]

\[ 58.73780530 \] (6.45)

**solve** \( V = 55, t \)

Warning, solving for expressions other than names or functions is not recommended.

Error, (in solve) a constant is invalid as a variable, 4.7

Because Maple is evaluating the names \( V \) and \( t \) in the \texttt{solve} operation, it is seeing \[ \texttt{solve} \left(35 + (65 - 35) \cdot \left(1 - e^{-\frac{4.7}{3}}\right), 4.7\right) \] which it cannot solve because there are no variables in the equation to solve for.

**unassign** \( t \)

**solve** \( V = 55, t \)

\[ 3 \ln(3) \] (6.47)

We can clear the path for solving by unassigning \( t \) first. Doing a \texttt{restart} would not work, because that would also unassign everything, including \( V \). We would lose the expression we want to solve for.
It can be tedious to have to remember to unassign variables if we want to go back to using them as symbols in the expression. We recommend using the `eval` operation (also available in the clickable menu as \( f(x) \bigg|_{x=a} \)) instead of assignment, if you are switching back and forth between using values for a variable and using it as a symbol. `eval` returns the same result as if you had done the evaluation, but the evaluation is automatically undone after the calculation is performed.

You can evaluate using values for several variables by giving a list of equations instead of a single equation as the second argument to `eval`.

<table>
<thead>
<tr>
<th>Evaluating an expression using a particular value of one of the variables in the expression using eval, and then solving</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>restart</strong></td>
</tr>
<tr>
<td>( V := 35 + (65 - 35) \cdot \left( 1 - e^{- \frac{t}{3}} \right) )</td>
</tr>
<tr>
<td>( V := 65 - 30 \cdot e^{- \frac{t}{3}} ) (6.48)</td>
</tr>
<tr>
<td><code>eval(V, t = 2.5)</code></td>
</tr>
<tr>
<td>51.96205374 (6.49)</td>
</tr>
<tr>
<td><strong>solve(V = 55, t)</strong></td>
</tr>
<tr>
<td>3 ( \ln(3) ) (6.50)</td>
</tr>
<tr>
<td><strong>t</strong></td>
</tr>
<tr>
<td>( t ) (6.51)</td>
</tr>
<tr>
<td><strong>Note that the <code>eval</code> did not assign a value to ( t ). It's still just a symbol at this point.</strong></td>
</tr>
</tbody>
</table>

**Begin parameters**

<table>
<thead>
<tr>
<th>( V_i := 35 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>35 (6.52)</td>
</tr>
<tr>
<td>( V_{\text{max}} := 65 )</td>
</tr>
<tr>
<td>65 (6.53)</td>
</tr>
<tr>
<td>( V_t := 55 )</td>
</tr>
<tr>
<td>55 (6.54)</td>
</tr>
<tr>
<td>( t_0 := 4.7 )</td>
</tr>
<tr>
<td>4.7 (6.55)</td>
</tr>
</tbody>
</table>

**End parameters**

| \( V_{\text{prime}} := V_i + (V_{\text{max}} - V_i) \cdot \left( 1 - e^{- \frac{t}{V_t}} \right) \) |

This is an example of how to set up a script using parameters while taking advantage of `eval`. Several symbols in the expression for voltage are set up and assigned as parameters. The expression that describes how voltage changes over time is not a parameter, although it is assigned a name for easier use in subsequent steps of the computation.
An example using assignment and eval

**restart**

**Begin parameters**

\[ V_i := 35 \]  
\[ V_{\text{max}} := 65 \]  
\[ V_t := 55 \]  
\[ t_0 := 4.7 \]

**End parameters**

\[ V' := V_i + (V_{\text{max}} - V_i) \cdot \left(1 - e^{-\frac{t}{\tau}}\right) \]

\[ V' := 65 - 30 e^{-\frac{t}{\tau}} \]  
\[ \tanExpr := \text{eval}(V', [t = t_0]) \]

This is an example of how to set up a script using parameters while taking advantage of `eval`. Several symbols in the expression for voltage are set up and assigned as parameters. But we use `eval` to maintain \( \tau \) and \( t \) as symbols in the expression \( V' \).

We use the information that the capacitor is observed to at \( V_t \) volts at time \( t_0 \) to find the value of \( \tau \) that is consistent with this.
\[ \text{tauValue} := \text{solve}(\text{tauExpr} = \text{v}, \text{tau}) \]
\[ \text{tauValue} := 4.278124365 \] (6.67)

\[ t\text{Expr} := \text{eval}(V\prime, \text{tau} = \text{tauValue}) \]
\[ t\text{Expr} := 65 - 30 \cdot e^{-0.2337472955 t} \] (6.68)

\[ \text{solve}(t\text{Expr} = \text{vt}, t) \]
\[ 4.699999999 \] (6.69)

\[ t\text{Expr} \] is the value of the expression with the values of the parameters and the calculated value of \( \tau \) plugged into \( V\prime \). We can solve an equation based on this formula to find the time when we achieve a voltage of 55 volts in this configuration.

6.11 Summary of Chapter 6 material

Troubleshooting textual input in Maple

<table>
<thead>
<tr>
<th>Remember to...</th>
<th>Examples with error(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Supply a function name and arguments (parameters)</td>
<td>solve ( x = 3 \cdot x^2 - 2 ) should be ( \text{solve}(x = 3 \cdot x^2 - 2, x) )</td>
</tr>
<tr>
<td>Match delimiters (parenthesis and brackets)</td>
<td>solve ( x = 3 \cdot x^2 - 2, x ) should be ( \text{solve}(x = 3 \cdot x^2 - 2, x) )</td>
</tr>
<tr>
<td>Press the right arrow key to exit from variable exponents</td>
<td>solve ( x = 3 \cdot x^2 - 2, x ) should be ( \text{solve}(x = 3 \cdot x^2 = 2, x) )</td>
</tr>
<tr>
<td>Set ranges correctly when plotting</td>
<td>( p\text{lot}(x - 3 \cdot x^2 - 2, x, -3.3) ) should be ( \text{plot}(x - 3 \cdot x^2 - 2, x = -3.3) )</td>
</tr>
</tbody>
</table>

Plotting examples

<table>
<thead>
<tr>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plotting multiple expressions</td>
</tr>
</tbody>
</table>
| Plotting with lists | \( x\text{List} := [1, 2, 3, 4] \)  
\[ x\text{List} := [1, 2, 3, 4] \] (6.70)  
\( y\text{List} := [5, 6, 7, -1] \)  
\[ y\text{List} := [5, 6, 7, -1] \] (6.71)  
\( \text{plot}(x\text{List}, y\text{List}, \text{style} = \text{point}) \) |
| Using multiple colors in a multi-plot | \( \text{plot}([\sin(x), \sin(\frac{x}{2}), \sin(2\cdot x)], x = -\pi .. \pi, 4-Pi, \)  
\text{color} = ["red", "green", "blue"] \) |
| Set the titles of the axes | \( \text{plot}(3.5 \cdot x^2 - 2, x = 1..5, \text{labels} = ["temperature (in degrees C)", "pressure in kilopascals"]) \) |

Using Maple's built-in help

<table>
<thead>
<tr>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Use Help&gt;Maple Help or press Ctrl-F1 (Command-F1 on a Mac) Remember that you can click on related topics when viewing the help for a particular command</td>
</tr>
</tbody>
</table>

Operations on lists examples

<table>
<thead>
<tr>
<th>Examples</th>
</tr>
</thead>
</table>
| Create a list | \( s\text{l} := [a, b, c, a] \)  
\[ [a, b, c, a] \] (6.72) |
### Specify a sublist of values

\[ ts1 := s1[1..3] \]
\[ [a, b, c] \]  
(6.73)

### Specify one item from the list

\[ s1[1] \]
\[ a \]  
(6.74)

### Specify a sublist with one item

\[ s2[3..3] \]
\[ [97] \]  
(6.75)

### Count the number of items in the list

\[ n := nops(s2) \]
\[ 5 \]  
(6.76)

### Add together all the items in the list

\[ \sum_{i=1}^{n} s2[i] \]
\[ 97.67 \]  
(6.77)

### Compute the average of all the numbers in the list.

\[ \frac{\sum_{i=1}^{n} s2[i]}{n} \]
\[ \frac{97.67}{5} = 19.534 \]  
(6.78)

### Convert a list into a sequence

\[ op(ts1) \]
\[ a, b, c \]  
(6.79)

### Convert a list into a string

\[ convert(s2, \text{string}) \]
\[ "[1, 3.47, 97, -5.9, 2.1]" \]  
(6.80)

### Solving a system of equations using lists

Create the system of equations

\[ \text{sys} := \{3 \cdot x + 5 \cdot y = 6, 2 \cdot x - 5 = y\} \]
\[ [3x + 5y = 6, 2x - 5 = y] \]  
(6.81)

Set the variables of the system

\[ \text{vars} := \{x, y\} \]
\[ \{x, y\} \]  
(6.82)

Solve the system and extract the first solution of possibly many solutions

\[ \text{sols} := \text{solve(sys, vars)}[1] \]
\[ \{x = \frac{31}{13}, y = -\frac{3}{13}\} \]  
(6.83)

### Evaluating an expression with eval instead of assignment

\textit{eval} allows you to substitute a value for a variable within an expression, without assigning that value to the variable.

\[ V := 35 + (65 - 35) \cdot \left(1 - e^{-\frac{t}{3}}\right) \]
\[ V := 65 - 30 e^{-\frac{t}{3}} \]  
(6.84)
<table>
<thead>
<tr>
<th>$eval(V, t = 2.5)$</th>
<th>51.96205374</th>
<th>(6.85)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$t$</td>
<td>(6.86)</td>
</tr>
</tbody>
</table>
7 Chapter 7 Using and Defining Functions

7.1 Chapter overview

Functions occur so much in mathematics that it's natural that Maple knows a lot about them and how to compute with them.

You can also define your own functions.

There are a few pitfalls in the use of functions in Maple:

a) The names of common mathematical functions used in Maple may differ from what you are used to.

b) Some functions in Maple do non-mathematical things, such as solve, and plot. Others take novel arguments -- lists, equations, and ranges, rather than numbers.

c) The way functions are defined uses := and -> rather than the use of = as found in math textbooks. This is because the "context-free" language processing of Maple thinks an equation is being defined whenever it sees an equal sign. It would be difficult for standard computer language-processing technology to use context to determine that an "equals" means "function definition".

7.2 Functions in computer languages: a way of producing an output from inputs

Everyone is introduced to the idea of a function in secondary school mathematics: Calculators can compute many of the common functions found in high school algebra and pre-calculus: sin, cos, ln, \( \sqrt{} \), etc.

Maple can evaluate these functions. The common ones are found in the Expression palette but there are hundreds more.

When entering a functional expression, the syntax used is:

`function name ( sequence of arguments )`

The parentheses are **mandatory** in Maple. Unexpected results, possibly including an error message may result if you forget them.

### Table 7.1: Function results in Maple

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin(.35), cos\left(\frac{3.14159265}{2}\right), \ln(3.72), \sqrt{2.0}</td>
<td>All the common functions know how to give limited-precision results given limited precision inputs (7.1)</td>
</tr>
<tr>
<td>cos\left(\frac{\pi}{2}\right), \ln(1), \sqrt{\frac{1}{2}}, \sin\left(\frac{35}{100}\right), \left(\frac{5}{2}\right), \log_{10}(1001), \log_{10}(1001.0)</td>
<td>Maple's programming will return an exact number if the function has that kind of result for the given input. However, if a simple exact result can't be found, Maple will return what you typed in. Sometimes there is a little simplification that goes on so what comes out is not literally what comes in, although it usually obvious that it is mathematically equivalent. For example, Maple will always simplify fractions by eliminating the greatest common divisor from the numerator and denominator. (7.2)</td>
</tr>
<tr>
<td>cos([a, b, c])</td>
<td>Error, invalid input: cos expects its 1st argument, x, to be of type algebraic, but received [a, b, c]</td>
</tr>
<tr>
<td>cos(2, \pi)</td>
<td>Error, (in cos) expecting 1 argument, got 2</td>
</tr>
</tbody>
</table>

This doesn't work because cosine expects an algebraic expression, not a list.

This mistake might come about if you typed a comma instead of a *.
What was this person thinking? Whatever it was, Maple doesn't know what to do with it.

Another delimiter message. Look for extra or missing parentheses.

This is what happens if you forget the mandatory parentheses. There is no error message, but what Maple is giving you is the product of Pi, the symbol "cos", and 1/4.

This is the correct way to compute the cosine of \( \frac{\pi}{4} \) radians.

7.3 The textual names of common functions: doing math calculations using the keyboard.

Some built-in mathematical functions have textual names that are already quite familiar from their use in mathematics textbooks: \( \sin, \cos, \ln \). Some are used so often that the most convenient thing to do is to remember their names. \( \sqrt{\text{sqrt}}, \text{abs}, \text{min}, \text{max} \) are straightforward -- they are naturally thought of as functions and have names that are abbreviations of the standard nomenclature. Others, such as \( \text{arcsin}, \text{log10} \) have names that make sense but you'd have to look them up in the Maple on-line documentation to know.

Example 7.2
\[ \cos \frac{\pi}{2} \]
Commentary: What was this person thinking? Whatever it was, Maple doesn't know what to do with it.

Example 7.3
\[ \cos \left( \frac{\pi}{4} \right) \]
Commentary: Another delimiter message. Look for extra or missing parentheses.

Example 7.4
\[ \cos \left( \frac{\pi}{4} \right) \]
Commentary: This is what happens if you forget the mandatory parentheses. There is no error message, but what Maple is giving you is the product of Pi, the symbol "cos", and 1/4.

Example 7.5
\[ \cos \left( \frac{\pi}{4} \right) \]
Commentary: This is the correct way to compute the cosine of \( \frac{\pi}{4} \) radians.

The attachment at the end of this chapter shows some of the many other functions available in Maple. Some of them work on lists, sets, equations rather than on numbers or expressions. However, the same principle applies: they have a rule for taking the value of their inputs (also known as arguments) and computing a result from them.
or the range or labels of the plot, it's easier to edit the text and re-execute the region than it would be with the clickable interface.

max( -3, 92, 43.7, 0, sqrt(16))

\[ 92 \] (7.8)

min(5, 7, 29, x, 3)

\[ \text{min}(3, x) \] (7.9)

\[
\max\left( \text{abs}(x), \frac{x}{2}, 5.7, \text{abs}(x) - 2, 0 \right)
\]

\[ \max\left( 5.7, \frac{1}{2} x, |x| \right) \] (7.10)

### 7.4 A function name to commit to memory: exp

The only one function whose textual usage may take getting used to is the exponential function \( \text{exp} \). Instead of writing \( e^x \), use \( \exp(x) \). The textual doppelganger \( e^x \) does not work as a way of calculating a power of \( e \), the base of the natural logarithm (where "\( e \)" is just the letter typed at the keyboard, not augmented by command completion as described in section XX). The orientation in college-level mathematics to view "a power of \( e \)" as a function is a pervasive change in point of view from what you may have seen in high school. Making a point to use the new notation frequently is the best way to make the switch.

exp is the name of the exponential function

\[
\exp(x) \cdot \exp(y)
\]

\[ e^x e^y \] (7.11)

\[
\text{assuming real}
\]

\[ e^{x+y} \] (7.12)

exp(x) is the textual way of writing "the symbolic constant \( e \) raised to the power \( x \). Maple knows how to simplify symbolic expressions with the simplify → assuming real operations in the clickable menu.
We can calculate powers of \( e \) from the keyboard. For comparison we do the same calculation using the Common Symbols and Expressions Palette. While more of a "sure thing", proficient users would be able to get the keyboard version entered more quickly.

\[
\exp(1) = 1.105170918 \quad (7.13)
\]

\[
e^1 = 1.105170918 \quad (7.14)
\]

We can enter an expression involving exponentials using only the keyboard. Maple will regard it as meaning the same thing as the expression entered using the combination of the keyboard and the Palette.

\[
V_i + (V_{\text{max}} - V_i) \left( 1 - \exp \left( -\frac{r}{\tau} \right) \right) = V_i + (V_{\text{max}} - V_i) \left( 1 - e^{-\frac{r}{\tau}} \right) \quad (7.15)
\]

### 7.5 How can I remember so many functions?

A well-developed system such as Maple, Matlab, or C# has thousands of built-in functions. It is unreasonable to expect that you can get the full gamut of professional work done knowing only three or four functions. The bad news is that you will have to remember at least the names of a several dozen functions. The good news is that learning about functions is not that taxing -- if you own a scientific calculator you've already dealt with a situation where you can operate a dozen functions.

A reasonable stance to take is to be familiar (i.e. know "by heart") functions and symbols that you often use and to be adept at using documentation to look up the details of the ones that you need only occasionally. In exams and test about computer functions, you may be quizzed on the details of the most common, but things rarely end up in a place where your grade will depend on how well you memorize thick lists of names.

A quick-recall method for access to common functions with textual entry is to type the escape key (Esc) after typing a few characters. A pop-up menu will appear that will list possible ways of completing what you typed. This is called command completion. Recall that you've already used this feature to enter symbolic constants such as \( e \), the natural logarithm base. If what you are entering is a Maple operation such as \( \texttt{solve} \), command completion will provide a textual template to fill in the rest of the arguments. It will also provide the parentheses required when entering functional notation textually.

**Table 7.2: Command Completion**

<table>
<thead>
<tr>
<th>Text</th>
<th>We type ( \texttt{sol} ) and then type the ESC key. A pop up menu shows option, including several forms of ( \texttt{solve} ).</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="sol" /></td>
<td><img src="image" alt="sol" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Text</th>
<th>We are not compelled to have a second variable of ( x ), it's just short-hand reminder that the second argument is the variable and the first one is the</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="solve" /></td>
<td><img src="image" alt="solve" /></td>
</tr>
</tbody>
</table>
7.6 Be savvy about using on-line documentation

Experienced users refer to the on-line documentation to help remember details about functions. As has mentioned earlier, this is available through Help -> Maple Help menu feature (key shortcut: press the control key and then the F1 key). If you recall a phrase or a name of a function, you can type it into the search field and the on-line help system will, like Google, produce the pages it has about your text entry. You can then explore further by pressing links. Trying the examples typically given at the end of the description is a good way to get a form that you can use for your own purposes.

Using on-line help

We want to find information about how to use the inverse sine function in Maple. We start up on-line help and type in “inverse trigonometric” into the search field, then hit the “search” button. The page we see does tell us that it’s probably called "arcsin" but we’d like to see more. We see a link in the "see also" which we click on.

This uncovers more links. The one that says "invtrig" seems promising so we click on that link.
7.7 Defining your own functions with -> (arrow)

Maple allows you to define simple functions with the use of ->. The general form is

\[ \text{function name} := (\text{sequence of arguments}) \rightarrow \text{expression that describes result} \, . \]

These can be entered through the Expression Palette, or textually. The arrow is entered textually by typing a - and then a >, with no spaces separating them. We give an example of function definition in the following example.

Table 7.3: A function definition in a math textbook

Fruit flies are placed in a half-pint milk bottle with a banana (for food) and yeast plants (for food and to provide a stimulus to lay eggs). Suppose that the fruit fly population after \( t \) days is given by

\[
P(t) = \frac{230}{1 + 56.5 \cdot e^{-0.37 \cdot t}}
\]

(a) What is the population after five days?

(b) How long does it take for the population to reach 180?

Analyzing the text, we see that it defines a function named \( P \). It takes one input (argument), \( t \), and produces as output whatever you get from evaluating the expression \( \frac{230}{1 + 56.5 \cdot e^{-0.37 \cdot t}} \). Assuming that Maple understands the definition of \( P \) like it does a built-in function, then the description of what happens when you evaluate \( P(5) \) would be:

"Substitute 5 for wherever you see \( t \) in the expression \( \frac{230}{1 + 56.5 \cdot e^{-0.37 \cdot t}} \). This gives you

\[
\frac{230}{1 + 56.5 \cdot e^{-0.37 \cdot 5}}
\]

. Perform all the arithmetic and relevant simplifications in this expression and return that as the result of the function."
Even though 5 is an exact number, because there are limited precision numbers in the expression we expect that the result will be a limited-precision number. If the expression had only exact numbers in it, then the calculation would be done exactly.

What we would like to do is to tell Maple about the definition of $P$ and use it in our work.

We can do this through the clickable interface. We anticipate reuse of this for other days and population levels, and turn it into a parameterized script:

### Table 7.4: User-defined functions through the Expression Palette

<table>
<thead>
<tr>
<th>User-defined functions through the Expression Palette</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>restart</td>
<td>The general form is from the $f := a \rightarrow y$ in the Expression palette. We alter slots in the template to mention the specific function by name, the name of the input, and the expression that describes how to calculate the output. Note that while the math text said &quot;$P(t) = ...&quot; in Maple the definition goes &quot;$P := (t) -&gt; ...&quot;. It's a mistake to literally transcribe the math notation into Maple and expect it to mean &quot;function definition&quot;.</td>
</tr>
<tr>
<td>Start of parameters</td>
<td></td>
</tr>
<tr>
<td><code>numDays</code> := 5</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(7.16)</td>
</tr>
<tr>
<td><code>popLevel</code> := 180</td>
<td></td>
</tr>
<tr>
<td>180</td>
<td>(7.17)</td>
</tr>
<tr>
<td><strong>End of parameters</strong></td>
<td></td>
</tr>
<tr>
<td>$P := t \rightarrow \frac{230}{1 + 56.5 \cdot e^{-0.37 \cdot t}}$</td>
<td></td>
</tr>
<tr>
<td>$t \rightarrow \frac{230}{1 + 56.5 \cdot e^{(-1) \cdot 0.37 \cdot t}}$</td>
<td>(7.18)</td>
</tr>
<tr>
<td><strong>Compute the number of flies after <code>numDays</code> = 5 days.</strong></td>
<td>With this definition, we can compute $P(5)$ using the natural mathematical notation.</td>
</tr>
<tr>
<td>$P(numDays)$</td>
<td></td>
</tr>
<tr>
<td>23.27016688</td>
<td>(7.19)</td>
</tr>
<tr>
<td><strong>We plot the function to see how the population grows. This is not needed by the problem but it helps us understand the situation better.</strong></td>
<td>We can plot $P(t)$ like we would any other expression. One the options to plot (see plot options in on-line help) is the ability to specify the title of the graph by giving <code>title=string</code> as an additional argument</td>
</tr>
</tbody>
</table>
User-defined functions through the Expression Palette

<table>
<thead>
<tr>
<th>plot(P(t), t = 0 .. numDays, title = &quot;Fruit flies like a banana&quot;, labels = [&quot;t&quot;, &quot;# of flies&quot;])</th>
</tr>
</thead>
</table>

To the `plot` function. Note that we are not using the clickable interface to do plotting.

![Fruit flies like a banana](image)

We compute when the population reaches desired level by solving the equation $P(t) = \text{popLevel}$.

```
soln := solve(P(t) = \text{popLevel}, t)
```

14.36533644

In anticipation of using the value in later work, we assign it to the variable `soln`. We would then intend to take further steps using `soln`.

End of script

We could have defined the function textually just by typing $P := (t) \rightarrow \ldots$ followed by the rest of the expression instead of using the Expression Palette.

### 7.8 Using functional evaluation instead of using assignment or eval

The problem with assigning variables is that it may cause unwanted side-effects, such as when trying to `solve` or `plot` with expressions involving those variables. We have seen in the section on `eval` (page 79) that we can use `eval` instead of assignment, which saves us the chore of having to assign and then unassign variables so that they remain as symbols when solving or plotting. We can more succinctly avoid this problem by defining a function based on the expression, and then causing evaluation to occur with standard functional notation.

Table 7.5: Comparing evaluation using `eval` and functional notation

<table>
<thead>
<tr>
<th><code>restart</code></th>
</tr>
</thead>
</table>

We set up an expression and then evaluate it at $t=4.7$ seconds.

```
V := 35 + (65 - 35) \cdot \left(1 - \frac{t}{3}\right)
```

$V := 65 - 30 e^{\frac{t}{3}}$

(7.21)
We create a function of \( t \) using the arrow notation. We can then evaluate the function at any value of \( t \) that we want through the standard notation. This takes a little less typing than using `eval`.

\[
\begin{align*}
\text{eval}(V, t = 2.5) & \quad 51.96205374 \\
\text{eval}(V, t = -2.5) & \quad -4.02927673
\end{align*}
\]

\[
\begin{align*}
Vf & := (t) \rightarrow 35 + (65 - 35) \left(1 - e^{-\frac{t}{3}}\right) \\
Vf & := t \mapsto 65 - 30 e^{-\frac{t}{3}} \\
Vf(2.5) & = 51.96205374 \quad (7.25) \\
Vf(-2.5) & = -4.02927673 \quad (7.26)
\end{align*}
\]

We can plot the function \( Vf \) in a natural way. `plot` also has an abbreviated way of plotting functions -- just give the name of the function and the plotting range, and omit the name of the functional argument. You can read more about this in the on-line help for `plot`.

\[
\begin{align*}
\text{plot}(Vf(t), t = 0..30, title = "Voltage over time")
\end{align*}
\]
7.9 Troubleshooting function definitions

Function definition is another place where the standard notation in mathematics does not work in Maple. Recall that the technology that is standard in most computer language understanding systems needs to assign a unique meaning to input from the way it looks. Equations already use "=" so if you use f(x) = ... Maple will understand you to be talking about an equation, not a function definition. Use :=

Table 7.6: Troubleshooting function definitions

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>F := (m1, m2, r) \rightarrow \frac{g\cdot m1\cdot m2}{r^2}</td>
<td>This defines a function for the gravitational attraction force. m1, m2 and r are the arguments. g is another symbol used in the expression but it isn't included as an argument.</td>
</tr>
<tr>
<td>g := 6.673e-11</td>
<td>We define the value of g. We expect not to change this while using the function. We found the value by typing &quot;gravitation&quot; into Maple online help.</td>
</tr>
<tr>
<td>(7.27)</td>
<td></td>
</tr>
<tr>
<td>F(1, 2, 10)</td>
<td>This calculates the attraction force between a 1 kilogram and 2 kilogram mass that is 10 meters apart. The force is in units of Newtons.</td>
</tr>
<tr>
<td>1.334600000 10^{12}</td>
<td></td>
</tr>
<tr>
<td>(7.28)</td>
<td></td>
</tr>
<tr>
<td>Fbad := (m1, m2, r) \rightarrow \frac{g\cdot m1\cdot m2}{r^2}</td>
<td>This doesn't work at all. Do you see the difference between the definition of F and Fbad? One uses assignment := the other is either mistyped or mistaken.</td>
</tr>
<tr>
<td>Error, invalid operator parameter name</td>
<td>This tries to use &quot;=&quot; instead of &quot;\rightarrow&quot; to define the function FF. There is no error message, but we get an unexpected expression rather than the number we were expecting. Someone could use this gibberish in further work if they weren't able to discern that the result isn't numerical as they were expecting.</td>
</tr>
<tr>
<td>(7.29)</td>
<td></td>
</tr>
</tbody>
</table>
This illustrates an alternative way of defining a function, although it is not the form prescribed by these notes. Rather than "=" as would appear in a math textbook, the assignment operation "::=" is used instead. This produces the following pop-up:

Clicking "ok" to function definition will create the proper function definition, as the subsequent line of the computation indicates. We get the same result as with F.

### 7.10 Using functions from library packages, with

Although we have seen a number of built-in functions so far in Maple, there are several thousand more. Some of them are defined in your Maple program when it starts up. However, it is not done for most of the built-in functions. There are so many that if that were done for all of them, Maple would take a long time to start up and would require large amounts of memory even before you had done any work in it.

Most built-in functions are organized into collections called packages. The general way to access a function belonging to a package is through package[function]. The least squares function belongs to a package named CurveFitting, hence its full name is CurveFitting[LeastSquares].

The with(package name) operation in Maple will load all the functions in the specified package into Maple. After this operation, functions can be referred to with just their "short name", e.g. LeastSquares rather than CurveFitting[LeastSquares]. Doing a with(package) can save you typing if you expect to use a function, or several functions, from a package several times during a Maple session. Ending the line with a colon (:) will suppress printing of all the functions in the package that usually occurs.

<table>
<thead>
<tr>
<th>with</th>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>restart</td>
<td>$pData := [134.2, 142.5, 155.0, 159.8, 171.1, 184.2];$</td>
<td>The CurveFitting package has a number of functions for data fitting. One is called LeastSquares. However, LeastSquares(...) does nothing. Although there is no error message, this is a mistake -- we didn't load the package in with a &quot;with&quot; so the least squares data fitting function isn't known. We get the same kind of behavior as if we had typed in $f(1,2,3)$ with $f$ undefined -- Maple just spits back what we typed in.</td>
</tr>
</tbody>
</table>
Doing a "with" gives the names of all the functions in the package.

Once we do the with, LeastSquares works.

We can fit a line to temperature as a function of pressure.

7.11 Attachment: some built-in problem-solving functions

The functions in the Expression palette have the same name and work similarly to those described in math textbooks. The operations discussed in this attachment are also found in math textbooks, but they are usually not given function names. It may seem novel to you that the rules for solving equations, factoring polynomials, or plotting can be collected together and given a function name. Yet this way of writing about such actions allows us to combine mathematics and working on it. Thus solve, plot, factor, etc. are true functions -- they have names, they are invoked with arguments, and return results that can be assigned to a variable.

<table>
<thead>
<tr>
<th>Textual names of common operations in Maple</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Function</strong></td>
</tr>
<tr>
<td>solve an expression or an equation</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>solve an expression or an equation numerically</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Function</td>
</tr>
<tr>
<td>-------------------------------------------------------------------------</td>
</tr>
<tr>
<td><code>fsolve(a \cdot x^2 + b \cdot x + c = 0, x)</code></td>
</tr>
<tr>
<td>plot (x^2 - 7 \cdot x - 98, x = -20\ldots20)</td>
</tr>
<tr>
<td><code>plot([t^2 + 7 \cdot t, 4], t = -8\ldots2, color = [&quot;DodgerBlue&quot;, &quot;Purple&quot;], labels = [&quot;time&quot;, &quot;velocity&quot;])</code></td>
</tr>
<tr>
<td><code>eval(x^2 + y^2, x = 1)</code></td>
</tr>
<tr>
<td><code>eval(x^2 + y^2, [x = 1, y = \pi + 2])</code></td>
</tr>
<tr>
<td><code>rhs(x = 3 \cdot x^2 + 1)</code></td>
</tr>
<tr>
<td><code>lhs(x = 3 \cdot x^2 + 1)</code></td>
</tr>
</tbody>
</table>

The third example shows that Maple thinks that "x=0..10" is an equation even if it isn't one in the standard mathematical sense.
| approximate with a limited-precision number | evalf | \( \text{evalf}(\pi) \) & 3.141592654 \hspace{1cm} (7.50) \text{evalf} has an optional second argument. If it's not there, Maple will compute a 10 digit approximation. If the second argument provided is a positive integer, then Maple will compute that many digits. \( \text{evalf}(\pi, 20) \) & 3.1415926535897932385 \hspace{1cm} (7.51) \\ \( \text{evalf}\left(\sin\left(\frac{\pi}{10}\right), 15\right) \) & 0.309016994374947 \hspace{1cm} (7.52) |
| convert between units | convert | \( \text{convert}(36.0, \text{units, inches, meters}) \) & 0.9144000000 \hspace{1cm} (7.53) \text{convert} does many things. When it is given four arguments and the second argument it units, then it expects the first argument to be a number, and the third and fourth to be expressions describing the units being converted from and to. Note that the units can be ratios or products rather than just names. \( \text{convert}(0.011, \text{units, radians, degrees}) \) & 0.06302535745 \hspace{1cm} (7.54) \\ \( \text{convert}\left(19.47, \text{units, gallons \over hour}, \text{liters \over minute}\right) \) & 1.228366124 \hspace{1cm} (7.55) |

### 7.12 Summary of Chapter 7 material

<table>
<thead>
<tr>
<th>Common mathematical functions (see on-line help for index of functions)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin, \cos, \tan, \sec, \csc, \cot ) &amp; trigonometric functions</td>
</tr>
<tr>
<td>( \exp ) &amp; exponential function ( \exp(x) = e^x )</td>
</tr>
<tr>
<td>( \text{arcsin}, \text{arccos}, \text{arctan}, \text{arcsec}, \text{arccsc}, \text{arccot} ) &amp; inverse trigonometric functions</td>
</tr>
<tr>
<td>( \sinh, \cosh, \tanh, \text{...} ) &amp; hyperbolic trigonometric functions</td>
</tr>
<tr>
<td>( \ln, \log_{10}, \log_b ) &amp; logarithm, log base 10, log base ( b )</td>
</tr>
<tr>
<td>( \text{min, max} ) &amp; minimum and maximum</td>
</tr>
<tr>
<td>( \text{abs, sqrt} ) &amp; absolute value, square root</td>
</tr>
<tr>
<td>( \text{ceil, floor, trunc, frac} ) &amp; ceiling, floor, truncation, fractional part</td>
</tr>
</tbody>
</table>

### Command Completion

When entering math, type the first part of the name of the function and then hit the escape key. A pop-up window will appear with a list of all- \( \text{sol} \) (then hit the \text{ESC} key)
ternative competitions of what you typed. Pick one, and a template will automatically be entered for you. Edit the spots of the template to fit your situation.

This is a way to get the computer to automatically type the right kind of delimiters.

### Defining custom functions with the arrow (→) notation

Create a custom function by naming it and its parameters. A custom function defined using arrow notation is of the form:

\[
\text{FunctionName} := \text{ParameterName} \rightarrow \text{Function}
\]

In this case the function name is \( P \) and its sole parameter is \( t \). For multiple parameters, use the form:

\[
\text{FunctionName} := (\text{ParameterName}_1, \text{ParameterName}_2, \ldots) \rightarrow \text{Function}
\]

To call the function, use the form:

\[
\text{FunctionName}(\text{ParameterValues})
\]

To plot the function, set the range of an independent variable, and use it as a parameter to your custom function.

To solve the function for a particular value, use the solve function.

### Troubleshooting function definitions

**Error**

Forgetting to use the assignment (=) operator.

\[
F_{\text{bad}} = (m_1, m_2, r) \rightarrow \frac{g \cdot m_1 \cdot m_2}{r^2}
\]

Error, invalid operator parameter name

\[
F_{\text{bad}} = (m_1, m_2, r) \rightarrow \frac{g \cdot m_1 \cdot m_2}{r^2}
\]

Using the equality operator (=) instead of the arrow in function definition.

\[
F := (m_1, m_2, r) = \frac{g \cdot m_1 \cdot m_2}{r^2}
\]

\[
(m_1, m_2, r) = \frac{g \cdot m_1 \cdot m_2}{r^2}
\]
<table>
<thead>
<tr>
<th>Troubleshooting function definitions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Although technically not an error, defining a function without using the arrow notation is not prescribed by these notes.</td>
</tr>
</tbody>
</table>

\[ F_2(m_1, m_2, r) := \frac{g m_1 m_2}{r^2} \]

\[ (m_1, m_2, r) \rightarrow \frac{g m_1 m_2}{r^2} \] (7.64)
8 Chapter 8 Programming with functions

8.1 Chapter overview

Most computer languages regard functions or procedures as the "lego blocks" for building programs. Not only is it expected that there will be a lot of different kinds of blocks that will be provided, but that you will build things by putting them several of them together. The way functions are used in this way is through daisy chaining -- by making the output of one function the input of another. These chains can then be defined to be functions themselves. By defining a few chains and then using them together, powerful combinations of operations can be custom-built quickly for the user's needs.

Most computer languages extend the concept of functions to go beyond the numbers or formulas that "mathematical functions" provide. In Maple, as in most of languages, a function can return other kinds of results. It is fairly common in Maple and other languages to have functions that produce as output a list, an equation, or a string as a result. As we shall see, we can even return a Maple plot as the result of a function. In a symmetric fashion, it is possible for computer functions to have lists, equations, plots, or strings as inputs.

8.2 Designing functions from context

In doing technical work, we often see functions defined as an equation relating the name of the function, its argument(s), and the function definition. Those are easy to translate into Maple's notation and use. For example if we see in a mathematics book "define \( f(x) = x^2 + 2x - v0 \)" then we can just transcribe it into the Maple function notation: \( f := (x) \rightarrow x^2 + 2x - v0 \).

In word problems, we have to "read between the lines" and design the function. This requires answering the questions:

a) What will the inputs to the function be? Try to give symbolic names for it.

b) What will the output be? Sometimes to realize what the output is, you can create a worksheet with several steps. If there is only one final result, then that should be the output.

c) How do you calculate the output from the inputs? Hopefully, there's a simple formula that describes this.

We illustrate this process with an example:

**Designing a function for pressure/temperature problems**

Problem p. 180 in Introduction to Engineering by Jay Brockman, Wiley, 2009

The Ideal Gas Law, as stated in Introduction to Engineering, is:

\[
P \cdot V = n \cdot R \cdot T
\]

\[
P V = n RT \tag{8.1}
\]

where

- \( P \) is pressure in Pascals (Pa)
- \( V \) is volume in \( m^3 \).
- \( n \) is the amount of gas in moles (mol),
- \( T \) is the temperature in degrees K,
- \( R \) is the gas constant, approximately \( 8.31 [m^3 \cdot Pa] / [K \cdot mol] \).
We want to solve the following problem (actually, various versions of it):

**Problem**

We measure the temperature and pressure of a gas. It has a pressure of 100 $[kPa]$ and a temperature 473 degree Kelvin. Then cool it so it has a pressure of 89.8 $[kPa]$. What is its temperature?

**Finding the answer**

First, we do the mathematical thinking and informal calculation that allows us to build a function that will solve all problems of this type:

For a fixed cylinder volume, according to the Ideal Gas Law:

$$\frac{P_1}{T_1} = \frac{P_2}{T_2}, \text{ so } T_2 = \frac{P_2}{P_1} \cdot T_1 \text{, so }$$

$$T_2 = \frac{89.8}{100} \cdot 473 = 424.7540000$$

**Function design**

We see that the problem wants us to calculate a temperature $T_2$ given the atmospheric pressure $P_1$, the internal pressure $P_2$, and the first temperature, $T_1$.

The output is $T_2$, the inputs are $P_1$, $P_2$, and $T_1$. We are free to name the function anything we want since the problem statement doesn't name this. We decide to call it something that reminds us of the purpose.

$$secondTemp := (P1, P2, T1) \rightarrow \frac{P2}{P1} \cdot T1$$

$$\frac{P1, P2, T1}{P1} \rightarrow \frac{P2 T1}{P1}$$

(8.2)
Testing and troubleshooting the function

We see whether we get the intended result with the numbers we've already worked out. Note that if we hadn't done the analysis, we wouldn't have any way of testing what we designed.

\[ \text{secondTemp}(89.8, 100, 473) \]

526.7260579 \hspace{1cm} (8.3)

Oops, that isn't the same result. What did we do wrong? The formula 1.2.2 seems like the right thing. What else could go wrong? Close inspection indicates that the first argument to \text{internalTemp} is P1, which appears in the denominator of the formula for the output. In (1.2.3), that would put the "89.8" in the denominator, but our example had 89.8 in the numerator. Oops, we gave the values in the wrong order for the function. There's nothing wrong except that we should invoke the function with the information given in the correct order:

\[ \text{secondTemp}(100, 89.8, 473) \]

424.7540000 \hspace{1cm} (8.4)

Using the function

We are given a different version of the problem:

We measure the temperature and pressure of a gas. It has a pressure of 2000 \text{ [kPa]} and a temperature 473 degree Kelvin. Then cool it so it has a pressure of 53.6 \text{ [kPa]}. What is its temperature?.

Answer:

\[ \text{secondTemp}(2000, 56.6, 473) \]

13.38590000 \hspace{1cm} (8.5)

Since the answer is in degrees Kelvin, this is only about 14 degrees above absolute zero. That's pretty cold!

The usefulness of alternative function designs

Suppose we had this new problem:

We measure the temperature and pressure of a gas. It has a pressure of 100 \text{ [kPa]} and a temperature 473 degree \text{ [K]}. We then heat it to 512 degrees Kelvin. What is its pressure then?

Another function designed

A little thought produces the calculation:

\[ P_2 = \frac{P_1}{\frac{T_1}{T_2}} = \frac{100}{\frac{473}{512.0}} = 108.2452431 \]
This leads to the function definition:

\[ \text{secondPressure} := (P1, T1, T2) \rightarrow \frac{P1}{T1} \frac{T2}{T2} \]

\[ (P1, T1, T2) \rightarrow \frac{P1T2}{T1} \quad (8.6) \]

We test this (remembering what happened before about the order of arguments)

\[ \text{secondPressure}(100, 473, 512.0) \]

\[ 108.2452431 \quad (8.7) \]

**Conclusion**

To develop functions, it helps to have worked through some typical calculations interactively. Once you have realized which quantities you are starting with and named them, and have developed the formula for the calculation using those names, you can create a function definition. You can use the names given in the problem description, or you can make up names based on their purpose. Unlike mathematics, you are not limited to single letters for names of variables or names of functions. Computer programmers know that longer names are often easier to remember or understand.

### 8.3 Function composition: daisy-chaining functions together

In the scripts we have developed so far, we have developed a result through a sequences of actions. These sequences can often be described through functional composition — an expression that chains together several actions. Consider the following example:

**Problem**

On November 1, 2007, one Euro was worth 1.002908434 US dollars. We are buying widgets that cost 30 Euros each and importing them into the US. We then put the widgets into packages that cost .075 dollars. How much should we sell the product in the US if we want a 10% markup from our costs?
Finding the solution: step 1, first do the calculation interactively.

Doing this in the style of scripts, we first assign the values to variables, and then do the calculational steps.

\[\text{convRate} := 1.002908434\]

\[1.002908434\]  \hspace{1cm} (8.8)

\[\text{costInEuros} := 30\]

\[30\]  \hspace{1cm} (8.9)

\[\text{pkgCost} := 0.075\]

\[0.075\]  \hspace{1cm} (8.10)

\[\text{markupPct} := 0.10\]

\[0.10\]  \hspace{1cm} (8.11)

\[\text{totalCost} := \text{pkgCost} + \text{convRate} \times \text{costInEuros}\]

\[30.16225302\]  \hspace{1cm} (8.12)

\[\text{sellingPrice} := (1 + \text{markupPct}) \times \text{totalCost}\]

\[33.17847832\]  \hspace{1cm} (8.13)

We foresee using this calculation several times as the conversion rate, the manufacturing cost in Europe, and the packaging cost change. We even see that the markup might change. We can try to boil down these steps into a few functions.

Designing the solution: step 2, design functions to do the calculational steps

\[\text{totalCostFunc} := (\text{convRate}, \text{costInEuros}, \text{pkgCost}) \rightarrow (\text{pkgCost} + \text{convRate} \times \text{costInEuros})\]

\[(\text{convRate}, \text{costInEuros}, \text{pkgCost}) \rightarrow \text{pkgCost} + \text{convRate} \times \text{costInEuros}\]  \hspace{1cm} (8.14)

\[\text{priceFunc} := (\text{markupPct}, \text{totalCost}) \rightarrow (1 + \text{markupPct}) \times \text{totalCost}\]

\[(\text{markupPct}, \text{totalCost}) \rightarrow (1 + \text{markupPct}) \times \text{totalCost}\]  \hspace{1cm} (8.15)

\[\text{sellingPriceFunc} := (\text{convRate}, \text{costInEuros}, \text{pkgCost}, \text{markupPct}) \rightarrow \text{priceFunc}(\text{markupPct}, \text{totalCostFunc}(\text{convRate}, \text{costInEuros}, \text{pkgCost}))\]

\[(\text{convRate}, \text{costInEuros}, \text{pkgCost}, \text{markupPct}) \rightarrow \text{priceFunc}(\text{markupPct}, \text{totalCostFunc}(\text{convRate}, \text{costInEuros}, \text{pkgCost}))\]  \hspace{1cm} (8.16)

Note that the way that the third function \text{sellingPriceFunc} is defined, it takes the output of \text{totalCostFunc} and makes it one of the inputs to \text{priceFunc}. 
Testing the solution: step 3, test the building blocks in the order that they are used

We test the first two functions. After we see that they agree with our preliminary version of the calculations, we test the third function that depends on the correctness of the first two.

\[
\text{totalCostFunc}(\text{convRate}, \text{costInEuros}, \text{pkgCost})
\]

\[30.16225302\]

\[\text{priceFunc}(0.10, (1.3, 10))\]

\[33.17847832\]

\[
\text{sellingPriceFunc}(\text{convRate}, \text{costInEuros}, \text{pkgCost}, \text{markupPct})
\]

\[33.17847832\]

We could write a script that just used \text{totalCostFunc} and \text{priceFunc}, but by designing and using a third function, we reduce the work of handling an instance of the problem to just pasting in the values for the four parameters in one line. This is probably less work than changing four lines of parameters that we have had with our previous approach to scripts.

While function composition is a succinct way of ordering many operations, its advantages are apparent only after the chain is built and tested as working correctly. It maybe easier to develop the chain as a script of assignments and then refactor -- rewrite without changing the meaning -- the script so that it uses user-defined functions to replace some of the chains of assignments.

Using the solution: step 4, present a script that defines the functions, then invoke the "answer function" repeatedly to handle various versions of the problem

A script that uses functional composition (chaining), and its use

Begin function definitions

\[
\text{totalCostFunc} := (\text{convRate}, \text{costInEuros}, \text{pkgCost}) \to (\text{pkgCost} + \text{convRate} \cdot \text{costInEuros})
\]

\[
(\text{convRate}, \text{costInEuros}, \text{pkgCost}) \to \text{pkgCost} + \text{convRate} \cdot \text{costInEuros}
\]

\[\text{(8.20)}\]

\[
\text{priceFunc} := (\text{markupPct}, \text{totalCost}) \to (1 + \text{markupPct}) \cdot \text{totalCost}
\]

\[
(\text{markupPct}, \text{totalCost}) \to (1 + \text{markupPct}) \cdot \text{totalCost}
\]

\[\text{(8.21)}\]

\[
\text{sellingPriceFunc} := (\text{convRate}, \text{costInEuros}, \text{pkgCost}, \text{markupPct}) \to \text{priceFunc}(\text{markupPct}, \text{totalCostFunc}(\text{convRate}, \text{costInEuros}, \text{pkgCost}))
\]

\[
(\text{convRate}, \text{costInEuros}, \text{pkgCost}, \text{markupPct}) \to \text{priceFunc}(\text{markupPct}, \text{totalCostFunc}(\text{convRate},\text{costInEuros},\text{pkgCost}))
\]

\[\text{(8.22)}\]

End function definitions

Problem solving

Version 1

On November 1, 2002, one Euro was worth \[\frac{1}{1.002908434} = 0.9971\] US dollars. We are buying widgets that cost 30 Euros each and importing them into the US. We then put the widgets into packages that cost .075 dollars. How much should we sell the product in the US if we want a 10% markup from our costs?
A script that uses functional composition (chaining), and its use

\[ \textit{sellingPriceFunc}(1.002908434, 30, .075, .10) \]

\[ \$33.18 \]  

(We got the number formatted to currency by right-click->Numeric Formatting->Currency.)

**Version 2**

On November 1, 2007, one Euro was worth 1.4487 US dollars. We are buying widgets that cost 33 Euros each and importing them into the US. We then put the widgets into packages that cost .09 dollars. How much should we sell the product in the US if we want a 10% markup from our costs?

\[ \textit{sellingPriceFunc}(1.4467, 33, .09, .10) \]

\[ \$52.61 \]  

**Version 3**

On November 1, 2009, one Euro was worth 1.4728 US dollars. We are buying widgets that cost 35 Euros each and importing them into the US. We then put the widgets into packages that cost .10 dollars. How much should we sell the product in the US if we want a 10% markup from our costs?

\[ \textit{sellingPriceFunc}(1.4728, 35, .10, .10) \]

\[ \$56.81 \]

We see that if we are interested in looking at the solution to several different versions of the problem, setting up the script as a collection of function definitions presents the problem-solving calculation only once. We can then proceed and present the several solutions through a single-line calculation. We don't have to wade through all the steps of the calculation to see the answer to the first problem, then looking through the same steps to see the answer to the second, etc.

### 8.4 Expressions with units of measurements: convert

Maple has facilities for converting between various English and metric units. It is useful for doing multi-step calculations because the conversions happen automatically.

In the first way of using \textit{convert}, one thinks of a value as implicitly expressing a number of units and wants another number expressing those number of units converted to another unit. One uses \textit{convert(value, units, fromUnit, toUnit)}.

<table>
<thead>
<tr>
<th>Examples of unit conversion</th>
<th>Note that the answer to this was expressed as a floating point number rather than a fraction because the input was floating point (1.0)</th>
</tr>
</thead>
</table>
| \textit{convert(1, units, inch, meter)} | \[
\begin{array}{c}
127 \\
5000
\end{array}
\]  

(8.26) |
| \textit{convert(1, units, ft, km)} | \[
\begin{array}{c}
381 \\
1250000
\end{array}
\]  

(8.27) |
| \textit{convert(1.0, units, mile, mm)} | \[
1.6093440 \times 10^6
\]  

(8.28) |
Examples of unit conversion

<table>
<thead>
<tr>
<th>Command</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>convert(5.4, units, kilowatt, horsepower)</code></td>
<td>7.241519284</td>
</tr>
<tr>
<td><code>convert(2.0, units, angstroms, micrometers)</code></td>
<td>0.0002000000000</td>
</tr>
<tr>
<td>[\text{convert}\left(15.0, \text{units}, \frac{\text{miles}}{\text{hour}}, \frac{\text{meters}}{\text{second}}\right)]</td>
<td>6.705600000</td>
</tr>
<tr>
<td>[\text{convert}\left(13.3, \text{units}, \frac{\text{gallons}}{\text{yard}^3}, \frac{\text{liters}}{\text{meter}^3}\right)]</td>
<td>65.85005144</td>
</tr>
</tbody>
</table>

Maple can convert between most compatible units.

Sometimes units are expressed as ratios of other units. Maple can handle such conversions as well.

In many examples in the Maple documentation, some of the arguments to `convert` are quoted -- surrounded by apostrophes -- to prevent evaluation from using the value of the names of the units. For example, if you have assigned a value to the variable `s`, then you cannot convert to seconds with this name without quotation.

Troubleshooting unit conversion

<table>
<thead>
<tr>
<th>Command</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{seconds} \leftarrow \text{convert}(3, \text{units}, \text{days, seconds}))</td>
<td>259200</td>
</tr>
<tr>
<td>(\text{seconds}^2 \leftarrow \text{convert}(4, \text{units}, \text{minutes, seconds}))</td>
<td>Error, (in convert/units) unable to convert ‘min’ to ‘259200’</td>
</tr>
<tr>
<td>(\text{seconds}^2 \leftarrow \text{convert}(4, \text{units}, \text{minutes, seconds}))</td>
<td>Quoting the 4th argument causes the name <code>seconds</code> to be given as the 4th input to <code>convert</code>. This works.</td>
</tr>
<tr>
<td>(\text{seconds}^3 \leftarrow \text{convert}(5, \text{units}, \text{hours, seconds}))</td>
<td>18000</td>
</tr>
</tbody>
</table>

As long as the various names used as arguments to the `convert` function don’t have values, things work fine.

Maple performs evaluation of names as it figures out what the inputs to `convert` is. Since `seconds` has a value, Maple tries to compute `convert(4, units, minutes, 259200)`. Since the 4th argument to `convert` has to be a name, an error results.

Quoting all the names as a prophylactic measure is acceptable. You see this in a lot of the Maple on-line documentation.

In Star Wars Episode IV: A New Hope, Han Solo says that the Millennium Falcon made the Kessel Run in “less than twelve parsecs”. We want to know how many days a parsec is.

\[\text{convert}(12.0, \text{units, parsecs, days})\] | Error, (in convert/units) unable to convert ‘pc’ to ‘d’ |
| \[\text{convert}(12.0, \text{units, parsecs, miles})\] | 2.300821388 \(10^{14}\) | (8.36) |

This is the error message you see when you are trying to convert between incompatible units, e.g. trying to convert a gallon into a meter. `pe` seems to be Maple’s internal name for parsec, `d` the name for days.

A parsec is a non-fictional unit of distance, not time, so we can convert 12 parsecs to miles, kilometers, inches... But we can't convert it to days any more than we can convert inches to volts.

A problem solved, a script built using function definitions

A car travels a 45 miles per hour. How many minutes does it take to travel 900 kilometers?

We build a sequence of calculations to understand how to solve this problem. This is the informal phase of development, while we are trying to understand what to do. Once we have an idea, we start designing functions and testing them.
### A problem solved, a script built using function definitions

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( distance := 900.0 )</td>
<td>( 900.0 )</td>
</tr>
<tr>
<td>( speed := 45 )</td>
<td>( 45 )</td>
</tr>
<tr>
<td>( d := \text{convert}(distance, units, kilometers, miles) )</td>
<td>( 559.2340730 )</td>
</tr>
<tr>
<td>( t := \frac{d}{speed} )</td>
<td>( 12.42742384 )</td>
</tr>
<tr>
<td>( \text{convert}(t, units,'hours','minutes') )</td>
<td>( 745.6454304 )</td>
</tr>
</tbody>
</table>

It would pretty obvious how to make a script out of this to handle any problem of the form: A car travels \( speed \) miles per hour. How many minutes does it take to travel \( distance \) kilometers... With a few user defined functions, we can get the answer with less typing/cutting/pasting.

\( d\text{Convert} := (distance) \rightarrow \text{convert}(distance, units, kilometers, miles) \)  
\( distance \rightarrow \text{convert}(distance, units, kilometers, miles) \)  
\( d\text{Convert}(900) \)  
\( \frac{781250}{1397} \) 

\( t\text{Calc} := (d, speed) \rightarrow \frac{d}{speed} \)  
\( (d, speed) \rightarrow \frac{d}{speed} \)  
\( t\text{Calc}((1.4.18), 45) \)  
\( \frac{156250}{12573} \) 

\( t\text{Conv} := (t) \rightarrow \text{convert}(t, units,'hours','minutes') \)  
\( t \rightarrow \text{convert}(t, units,'hours','minutes') \)  
\( t\text{Conv}(1.4.15) \)  
\( \frac{3125000}{4191} \) 

The first step was to convert the distance from kilometers to miles. We create a function that does it. We test it to the result that we got in the script above and see that it agrees.

The next step was to calculate the time (in hours) from the distance in miles and the speed in mph. The test shows that \( t\text{Calc} \) seems to be built correctly.

The third step was to convert the time from hours to minutes. The test of this step agrees with what the script does, too.
### A problem solved, a script built using function definitions

\[
solveLt := (\text{speed}, \text{distance}) \\
\quad \rightarrow tConv(tCalc(dConvert(distance), \text{speed})) \\
\quad (\text{speed}, \text{distance}) \\
\quad \rightarrow tConv(tCalc(dConvert(distance), \text{speed})) \\
solveLt(45, 900.0) \\
745.6454304
\]  

The solution function chains together the three functions we've developed. This concludes the development and testing. We present a script and several solved problems in the figure below.

<table>
<thead>
<tr>
<th>Problem version A</th>
<th>Problem version B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A car travels at 45 miles per hour. How many minutes does it take to travel 900 kilometers?</td>
<td>A car travels at 45 miles per hour. How many minutes does it take to travel 452 kilometers?</td>
</tr>
<tr>
<td>( \text{travelSoln}(45, 900.0) )</td>
<td>( \text{travelSoln}(45, 452.0) )</td>
</tr>
<tr>
<td>745.6454304</td>
<td>374.4797052</td>
</tr>
</tbody>
</table>

The above table showed the thinking behind the design and testing of the multi-step calculation. However, in "what we would hand in", we don't show the testing or the initial script, just the definition of the functions, and then the repeated invocation of the "solution function“. Define the functions once, then invoke it repeatedly. This eliminates the need for repeated cutting/pasting/selection for execution.

### Solving multiple versions of a problem through functions

#### Begin function definitions

\[
dConvert := (\text{distance}) \rightarrow \text{convert(distance, units, kilometers, miles)} \\
distance \rightarrow \text{convert(distance, units, kilometers, miles)} \\
\]

\[
tCalc := (d, \text{speed}) \rightarrow \frac{d}{\text{speed}} \\
(d, \text{speed}) \rightarrow \frac{d}{\text{speed}} \\
\]

\[
tConv := (t) \rightarrow \text{convert(t, units, 'hours', 'minutes')} \\
t \rightarrow \text{convert(t, units, 'hours', 'minutes')} \\
\]

\[
\text{travelSoln} := (\text{speed}, \text{distance}) \\
\rightarrow tConv(tCalc(dConvert(distance), \text{speed})) \\
\quad (\text{speed}, \text{distance}) \\
\quad \rightarrow tConv(tCalc(dConvert(distance), \text{speed}))
\]

#### End of function definitions

The \text{travelSoln} function indicates the parameters of the script, \text{speed} and \text{distance}. In setting up the solution method as a function, we lose some intelligibility because of the "inside out" style of following the operation of daisy-chained function composition.

However, we gain convenience using the script. Using this form makes it easier to see several solutions, because you don't have to wade through all the lines of script that work out the answer, just a single line setting up the values of the parameters and printing out the answer.
Solving multiple versions of a problem through functions

Problem version C
A car travels at 65 miles per hour. How many minutes does it take to travel 1500 kilometers?

\[
\text{travel\text{So}hn}(65, 1500.0)
\]

\[860.3601126\] (8.56)

For casual unit conversion, it can still be useful to rely upon Maple's encyclopaedic knowledge of how to convert units. You can access this through Tools->Assistants->Unit Calculator

Table 8.2: Unit Converter Assistant

8.5 Inputs and outputs to user-defined functions don't have to be numbers

Although you don't see much mention of this in mathematics texts, it is fairly common while programming to define and use functions that take inputs and produce outputs that are not numbers. For example, if we have a list L of numbers, we can create a function that takes a list as input and produces the average of all the numbers as its output.

Table 8.3: A function that takes a list as its input

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
</table>
| \[
\text{average} := L \rightarrow \frac{\sum_{i=1}^{\text{nops}(L)} L[i]}{\text{nops}(L)}
\] | We are expecting the input to be a list of numbers. As explained in chapter 4, L[i] uses indexing to get the i-th value of the list. (See section 4.2.) nops(L) is the number of elements in the list. |
| \[\text{average}([5, 7,-3, 2, 6])\] | When we invoke the function, L is [5,7,-3,2,6], so nops(L) is 5. Since at least one of the elements of the list was a limited precision number (5. has a decimal point), the limited precision arithmetic is performed with it and subsequent steps of the sum. |
| \[3.375000000\] | (8.59) |
In analyzing mathematical models as we have been doing, it is also useful to produce abbreviations for common combinations of plot options by creating a function that produces a plot as its result.

**Table 8.4: A function that returns a plot as its output, rather than a number**

<table>
<thead>
<tr>
<th>A function that returns a plot as its output, rather than a number</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Problem</strong></td>
</tr>
<tr>
<td>We are given two lists of data, $pData$, and $tData$. Plot $pData$ as a function of $tData$, and vice versa.</td>
</tr>
<tr>
<td><strong>Solution</strong></td>
</tr>
<tr>
<td>Build a function that becomes an abbreviation for the operations in the plot. Provide a third argument that is the string for the color.</td>
</tr>
</tbody>
</table>

$$\text{PlotIt} := (xData, yData, c, L) \rightarrow \text{plot}(xData, yData, style = \text{point}, color = c, labels = L)$$

(8.60)

$$pData := [134.2, 142.5, 155.0, 159.8, 171.1, 184.2];$$

(8.61)

$$tData := [0, 20.1, 39.8, 60.0, 79.9, 100.3];$$

(8.62)

Plot pressure versus temperature, in red. Note: there seems to be a bug in Maple that suppresses the printing of the horizontal axis label.

$$\text{PlotIt}(pData, tData, \text{"red"}, [\text{"pressure"}, \text{"temperature"}])$$

Plot temperature versus pressure, in red. Copying and pasting the invocation of the PlotIt function we defined is easier than changing the insides of the original plot operation.
A function that returns a plot as its output, rather than a number

A function that returns a plot as its output, rather than a number

It is possible to return a list or sequence as a result of a function. Such a function can be put in a chain.

Table 8.5: A problem solved with a function that outputs a sequence of two numbers

A problem solved with a function that outputs a sequence of two numbers

Problem A

A right triangle has a hypotenuse of length 5 feet. The angle between it and one of its sides is 10 degrees. What is the length of the perimeter?

We first build a function that computes the two sides of the right triangle and returns the two values as a sequence. We have to convert degrees into radians in order to do this because the Maple trig functions all use radians.

\[ \text{sideSide} := (\text{hypo}, \text{angle}) \rightarrow (\text{hypo} - \text{sin}(\text{convert}(\text{angle} \cdot \text{degrees}, \text{radians})), \text{hypo} - \text{cos}(\text{convert}(\text{angle} \cdot \text{degrees}, \text{radians}))) \]  

(8.63)

Let's test the sideSide function.

\[ \text{sideSide}(5, 10.0) \]

5 \sin(0.05555555556 \pi), 5 \cos(0.05555555556 \pi) \]  

(8.64)

Now, develop a function that takes a sequence of three numbers and adds them together.

\[ \text{sumSides} := (a, b, c) \rightarrow a + b + c \]

(8.65)

By chaining together the output of sideSide and making it part of the input of sumSides, we can get the whole computation done in one function.
A problem solved with a function that outputs a sequence of two numbers

\[
\text{perimeter} := (\text{hypo}, \text{angle}) \rightarrow \text{sumSides}(\text{sideSide}((\text{hypo}, \text{angle}), \text{hypo})
\]

(8.66)

\[
\text{perimeter}(5, 10)
\]

\[
5 \sin\left(\frac{\pi}{18}\right) + 5 \cos\left(\frac{1}{18} \pi\right) + 5
\]

at 5 digits

\[10.792\]

(8.67)

(8.68)

Problem B

A right triangle has a hypotenuse of length 10 feet. The angle between it and one of its sides is 42 degrees. What is the length of the perimeter?

\[
\text{evalf}((\text{perimeter}(10, 42))
\]

\[24.12275432\]

(8.69)

Once we have done the work to design and test the functions out on a problem, we can present a script that can solve several different versions of the problem:

Solving several versions of a function with function definitions

Begin function definitions

A function that computes the two sides of a right triangle given the angle and the length of the hypotenuse

\[
\text{sideSide} := (\text{hypo}, \text{angle}) \rightarrow (\text{hypo} \cdot \sin(\text{convert(\text{angle} \cdot \text{degrees}, \text{radians}})), \text{hypo} \cdot \cos(\text{convert(\text{angle} \cdot \text{degrees}, \text{radians}}))
\]

(8.70)

A function that takes a sequence of three numbers and adds them together.

\[
\text{sumSides} := (a, b, c) \rightarrow a + b + c
\]

(8.71)

Compute the perimeter by summing the three sides.

\[
\text{perimeter} := (\text{hypo}, \text{angle}) \rightarrow \text{sumSides}(\text{sideSide}((\text{hypo}, \text{angle}), \text{hypo})
\]

(8.72)

End function definitions

Problem A Solution

A right triangle has a hypotenuse of length 5 feet. The angle between it and one of its sides is 10 degrees. Approximately, what is the length of the perimeter in feet?

\[
\text{evalf}((\text{perimeter}(5, 10))
\]

\[10.79227965\]

(8.73)
### Function design

| Designing functions from context | a) What will the inputs be?  
b) What will the output be?  
c) How do we calculate the output from the inputs? |

### Function composition

\[
A := (x, y) \rightarrow \frac{1}{x} + 3 \cdot x^2 + 3 \cdot y \\
(x, y) \rightarrow \frac{1}{x} + 3 \cdot x^2 + 3 \cdot y
\]

\[
B := (x, y) \rightarrow \frac{3}{A(x, y)} \\
(x, y) \rightarrow \frac{3}{A(x, y)}
\]

\[
B(3, 1) = \frac{9}{253}
\]

### Unit conversion

<table>
<thead>
<tr>
<th>Units can be converted directly into compatible units.</th>
<th>convert(1, units, inch, meter)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>convert(1, units, inch, meter)</td>
</tr>
<tr>
<td></td>
<td>127</td>
</tr>
<tr>
<td></td>
<td>5000</td>
</tr>
<tr>
<td>Compound units expressed in ratio form can also be converted into compatible compound units.</td>
<td>convert( \left( 15.0, \text{units, hours}^{-1}, \text{meters} \right) )</td>
</tr>
<tr>
<td></td>
<td>convert( \left( 15.0, \text{units, hours}^{-1}, \text{meters} \right) )</td>
</tr>
<tr>
<td></td>
<td>6.705600000</td>
</tr>
<tr>
<td>Converting between incompatible units will generate an error message.</td>
<td>convert(3, units, days, miles)</td>
</tr>
<tr>
<td></td>
<td>Error, (in convert/units) unable to convert ’d’ to ‘mi’</td>
</tr>
<tr>
<td>If we create a variable with the same name as a unit, trying to convert using the variable name will throw an error.</td>
<td>seconds := 5</td>
</tr>
<tr>
<td></td>
<td>seconds := 5</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
<tr>
<td>To avoid this, we can use single quotes around the unit names to specify using the unit string, as opposed to the variable value.</td>
<td>convert(3, units, days, seconds)</td>
</tr>
<tr>
<td></td>
<td>convert(3, units, days, seconds)</td>
</tr>
<tr>
<td></td>
<td>Error, (in convert/units) unable to convert ’d’ to ‘5’</td>
</tr>
</tbody>
</table>
### Unit conversion

```maple
can use Maple's built-in unit converter to convert units using drop-down menus.
```

### Non-number inputs and outputs of a function

Using a list of numbers as an input:

\[
\text{average} := L \rightarrow \frac{\sum_{i=1}^{nops(L)} L[i]}{nops(L)}
\]

\[
\text{average}([5, 7, -3, 2, 6])
\]

\[
\frac{17}{5}
\]

Returning a plot instead of a number:

\[
\text{PlotIt} := (xData, yData, c, L) \rightarrow \text{plot}(xData, yData, style = \text{point}, color = c, labels = L)
\]

\[
\text{PlotIt}([1, 2, 3], [2, 2, 1], \text{‘blue’}, ['x’, ‘y”])
\]

Returning a sequence of numbers instead of a single number:

\[
\text{sideSide} := (\text{hypo, angle}) \rightarrow (\text{hypo \cdot sin(convert(\text{angle degrees}, \text{radians}))),}
\]

\[
\text{hypo \cdot cos(convert(\text{angle degrees}, \text{radians}))})
\]

\[
\text{sideSide}(1, 45)
\]

\[
\frac{1}{2} \sqrt{2}, \frac{1}{2} \sqrt{2}
\]
Chapter 9 Visualization, modeling, and simulation

9.1 Chapter Overview

1. *Plot structures* are the result of the `plot` operation. They can be assigned to variables through `:=` just as numbers, formulas, lists, and function definitions, can be.

2. The `display` operation of the `plots` package allows you to combine together plots. Often times a single picture can be more enlightening or easier to understand than looking at multiple pictures separately.

3. *Simulation* is the art of predicting the behavior of system entities as they change over time, through the use of mathematical models. It can be as simple as using functions that, given the time `t` as input, calculate the position, size, weight, or other changing properties of a situation. With the appropriate mathematics, personal computer or supercomputer-class calculations can be used to come up with reasonably accurate descriptions of phenomena. Computational simulations have become a mainstay of modern engineering because the "build it and see" methodology often seen in elementary student work is not cost-effective once one moves about beyond simple scenarios that are inexpensive to test.

4. The `animate` operation of the `plots` package is explained. Its use is illustrated with a session of question-answering using a mathematical models of moving bodies. Computer-generated animations are another useful tool besides `solve`, and `plot`.

9.2 plot structures

Like `solve`, Maple plot is a function: it has inputs and produces outputs. What kind of output does the plot function produce? In Maple, the result of `plot` is a special type of result called a *plot structure*. When you evaluate an expression in Maple that invokes the plot function, a plot structure is created. If the plot structure is then assigned to a variable (through `:=`, for example), then an ellipsis of the plot structure is displayed. If the plot structure is the entire result and there is no assignment, then the plot is displayed.

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>plot(sin(x), x = 0..10)</code></td>
<td>The result of this operation is a plot structure. Since it is the entire result and is not being assigned to a variable through <code>:=</code>, Maple displays the structure in a pretty way -- by drawing a picture of all the points and labels established by the plot operation and put into the plot structure.</td>
</tr>
<tr>
<td><code>p := plot(cos(x), x = 0..10)</code></td>
<td>Evaluating an expression and then assigning it to the name <code>p</code>. This does not display the plot. We just see <code>PLOT(...)</code> which is a sign that the value of <code>p</code> is a plot structure.</td>
</tr>
</tbody>
</table>
Evaluating an expression -- just \( p \) -- causes the result (the plot) to be displayed.

The value of assigning a plot to a variable is explained in the next example, which uses another Maple operation, \texttt{display}, to combine plots together.

\begin{table}[h]
\centering
\begin{tabular}{|l|l|}
\hline
\textbf{Example} & \textbf{Commentary} \\
\hline
\( p \) & Evaluating an expression -- just \( p \) -- causes the result (the plot) to be displayed. The value of assigning a plot to a variable is explained in the next example, which uses another Maple operation, \texttt{display}, to combine plots together. \\
\hline
\end{tabular}
\end{table}

\section*{9.3 plots[\texttt{display}] and combining plots}

The display function from the plots package takes as its first argument a list of plot structures. It will produce a plot structure that combines all the plots together. \texttt{display} is the way to get a multi-plot in a script without doing cutting and pasting of plots.

\begin{table}[h]
\centering
\begin{tabular}{|l|l|}
\hline
\textbf{Table 9.2: Display combines plots} & \\
\hline
\texttt{display combines plots} & \\
\hline
\( \texttt{timeData} := [4.5, 5, 5.5, 6, 6.5, 7, 7.5, 8, 8.5, 9, 9.5, 10, 10.5, 11, 11.5, 12, 12.5, 13] \) & Someone we know has used a variant on LeastSquares Curve Fitting, to derive a formula with that fits the data. We wish to plot both the data points and the formula on a single graph. \\
\[ [4.5, 5, 5.5, 6, 6.5, 7, 7.5, 8, 8.5, 9, 9.5, 10, 10.5, 11, 11.5, 12, 12.5, 13] \] & \texttt{(9.2)} \\
\hline
\( \texttt{tempData} := [58, 57.5, 57., 56, 55.5, 54, 53, 52, 51, 50, 49, 48.5, 48.5, 47.5, 47.5, 46, 45.5, 45] \) & \\
\[ [58, 57.5, 57., 56, 55.5, 54, 53, 52, 51, 50, 49, 48.5, 48.5, 47.5, 47.5, 46, 45.5, 45] \] & \texttt{(9.3)} \\
\hline
\end{tabular}
\end{table}
### display combines plots

<table>
<thead>
<tr>
<th>$p1 := \text{plot}(\text{timeData}, \text{tempData}, \text{style = point, color = &quot;red&quot;})$</th>
<th>This is the point plot we get from this data.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p1$ is assigned the plot structure of the point plot. Because the result is assigned to the variable $p1$, only an ellipsis of the plot structure is displayed rather than the picture of the plot.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\text{formula} := 25.0 + 33.61425396 e^{-0.06172329065 t + 0.2777548079}$</th>
<th>Let's assume that we have a friend who has done curve fitting to $\text{timeData}$ and $\text{tempData}$ and gotten this formula. In ENGR 101 Fall 2010, Matlab was used to do exponential curve fitting like this. We could replication that calculation in Maple, but don't show it here. If you're interested in seeing how to do it, ask your instructor about it.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$25.0 + 33.61425396 e^{-0.06172329065 t + 0.2777548079}$</td>
<td>$p2$ is assigned another plot structure, the smooth plot of the formula. We skip the step where we show this plot as a separate picture because we are more interested in seeing it combined with the point plot.</td>
</tr>
<tr>
<td>$p2 := \text{plot}\left(\text{formula}, t = \min(\text{timeData}) \ldots \max(\text{timeData}), \text{color = &quot;blue&quot;} \right)$</td>
<td></td>
</tr>
</tbody>
</table>
9.4 plottools, lines, and other shapes

The plottools package has a number of functions that are useful for inclusion in visualizations (plots). For example, you can create a line segment of any desired color and line thickness with the `line` function.

One can then use the `display` function to merge together lines, plots, and other shapes.

The on-line documentation on plottools contains links to further description.

Table 9.3: plottools: lines and other shapes: a frivolous drawing

We create a bit of "modern art" by drawing a circle and a point plot.

```plaintext
with(plots):

[arc, arrow, circle, cone, cuboid, curve, cutin, cutout, cylinder, disk, dodecahedron, ellipse, ellipticArc,
  hemisphere, hexahedron, homothety, hyperbola, icosahedron, line, octahedron, parallelepiped,
  pie, point, polygon, project, rectangle, reflect, rotate, scale, semitorus, sphere, stellate,
  tetrahedron, torus, transform, translate]
```

As the on-line documentation indicates, the first two arguments to `line` are lists indicating the coordinates of the starting point and ending point of the line segment. There are optional arguments that indicate color and line thickness, etc.

```plaintext
I := line([[0, 1], [1, 1]], color = "orange);
    CURVES([[0, 1], [1, 1]], COLOUR(RGB, 0.80000000, 0.19607843, 0.19607843))

I2 := line([[1, 1], [1, 2]], color = "blue", thickness = 20)
    CURVES([[1, 1], [1, 2]], COLOUR(RGB, 0.0, 0.1.00000000, THICKNESS(20))
```
We suppress printing of the plot structure CURVES(...) for c3 with a colon because it's long and we don't want to see it, we want to see the picture it describes. Note that Maple exposes its Canadian roots by using "colour".

\[
c3 := \text{circle}([2, 2.5], 3, \text{color} = \text{"Purple"}, \text{thickness} = 5) : \\
\text{with(plots) :} \\
\text{display}([11, 12, c3, \text{plot}(2 \cdot r^2, t = 0 .. 2, \text{style} = \text{point}, \text{symbol} = \text{circle}, \text{symbolsize} = 30)])
\]

One of the options to plot (and display) is to not show the axes. Another option is to indicate that the scaling should be constrained to be equivalent in both horizontal and vertical directions (to make the circle look like a circle). That gives us an unframed work of art!

\[
\text{display}([11, 12, c3, \text{plot}(2 \cdot r^2, t = 0 .. 2, \text{style} = \text{point}, \text{symbol} = \text{circle}, \text{symbolsize} = 30)], \text{axes} = \text{none}, \\
\text{scaling} = \text{constrained})
\]

We can use lines and circles for less frivolous purposes, too.
We create a plot structure that is a green line segment running between the points (12,25), and (12, 58).

(9.10)

We can create a picture with the data plot and the exponential curve plot of the previous section, plots and the vertical green line. This highlights the value of the curve at t=12 minutes.

9.5 Mathematical models and simulation

Mathematical models try to describe a "real" situation in terms of equations and formulae. The point of modeling is to try, through mathematical or computational means, to determine what will happen without having to run experiments in the "real" situation.

As Dr. Jay Brockman of the University of Notre Dame says in his book *Introduction to Engineering*:

Some engineering students have been fortunate enough to participate in pre-engineering programs such as the first LEGO(TM) League robotics design competition or American Society of Civil Engineering bridge-building contests. In addition to fostering creative problem-solving skills, such projects also introduce students to the important notion that seemingly good ideas don't always work out in practice. Often in such programs, students have ample opportunity to test and modify their designs before they formally evaluate them. If the design doesn't work, then like a sculptor working with clay, the designer adds something here or removes something there until the design is acceptable.

This cut-and-try methodology is also sometimes used in industry, particularly in circumstances where the design is simple, or where, the risk or cost of failure is low. In many situations, however, there is no second chance in the event of failure. For engineering systems such as buildings, bridges, or airplanes -- top name just a few -- failure to meet specifications could mean a loss of life. For others -- such as the integrated circuit chip -- the cost of fabrication is so high that a company may not be able to afford a second chance. In these situations, it's critical for the engineering team to be highly confident that a design will be acceptable before it's built. To do this, engineers use *models* to predict the behavior of their designs. A model is an approximation to a real system, such that when actions are performed on the model, it will respond in a manner similar to the real system. Models can have many different forms, ranging from physical prototypes such as a crash-test dummy to complex computer simulations.

We can think of a mathematical model as a kind of virtual system... whose input is a set of variables that represent either aspects of the design or aspects of the environment, and whose output is a set of variables that represent the behavior of the system. Inside is a set of mathematical relationships that describe the operation of the system.
The point of expressing a situation mathematically is to use mathematics and computation to better understand the situation. Usually we are given or derive formulas that allow us to calculate key properties of the system. For models involving only a few variables, this can involve the following kinds of actions:

1. Get a single number, by evaluating a formula or function.
2. Get a single number, by solving an equation.
3. Gain an understanding of a relationship between one or more entities of interest and the "input variables" by producing a formula.
4. Gain a visual understanding of the relationship by plotting a function, or possibly several plots merged together.
5. Gain an understanding of how a system changes over time. Rather than computing the value of a variable once, we repeatedly compute the value of the variable at several different points at time. This is called \textit{computational simulation} of the system. We can view plots changing over time by producing an animation.

\begin{tabular}{|l|}
\hline
\textbf{Examples of the first four types of computation} \\
\hline
\textbf{Third type: producing a formula} \\
In the temperature-pressure data fitting example (page 99) the mathematical model is the formula that expresses the relationship between temperature and pressure. We had only data and no formula to begin with, but we developed the formula using the CurveFitting[LeastSquares] operation. This was a computation of the third type mentioned above. \\
\textbf{First type: Evaluating a formula or function} \\
Once we had the formula, we used the relationship to calculate pressure at several given temperatures. This was a computation of the first type mentioned above. \\
\textbf{Second type: solving an equation to get a desired value} \\
We also found a temperature corresponding to a specified pressure by using solve -- a computation of the second kind mentioned above. \\
\textbf{Fourth type: visualization (plotting)} \\
In the example with an exponential curve fit (page 122), we got a visual impression of how the formula fit the data by combining the point plot of the data and the plot of the formula together -- the fourth kind of computation mentioned above. \\
\hline
\end{tabular}

We haven't explained how to do the fifth type of computation -- animation -- yet. This will be discussed in upcoming section \textit{Animations (movies) using animate} (page 129)

### 9.6 Drawing x-y position as a function of time through parameterized plots

The mathematical models often describe the position of a system entity as a function(s) of time. If the entity's position is two dimensional, then we have two functions, often called $x(t)$ and $y(t)$. We can generate a plot of position for various values of $t$ with a special form of \texttt{plot}.

\begin{verbatim}
plot( [x-position expression, y-position expression, var = low..high], plot options)
\end{verbatim}

will draw a two dimensional graph connecting the $(x,y)$ points traced out for the values of the expression as the variable $\texttt{var}$ takes on values between $\texttt{low}$ and $\texttt{high}$.

\begin{tabular}{|l|}
\hline
\textbf{Examples of plots where the x and y positions are parameterized by t} \\
\hline
\textbf{Example 1} \\
In this example, we describe the $x$ and $y$ positions as periodic functions of time. Every $2\cdot\pi$ time units, the positions return back to where they were originally. \\
\hline
\end{tabular}
Examples of plots where the $x$ and $y$ positions are expressions parameterized by $t$

\[
xpos := (t) \rightarrow 3 \cdot \cos(t)
\]

\[t \rightarrow 3 \cos(t) \tag{9.11}\]

\[
ypos := (t) \rightarrow 2 \cdot \sin(t)
\]

\[t \rightarrow 2 \sin(t) \tag{9.12}\]

We plot the position using parameterized plotting. By default the scaling used for the horizontal and vertical axes are different, but in this case the default makes the graph look misleadingly like a circle when it should look more like an ellipse. To compensate, we use the `scaling=constrained` option to `plot`.

\[
\text{plot}([xpos(t), ypos(t), t = 0..2\cdot \text{Pi}], \text{scaling} = \text{constrained})
\]

Example 2

In this example, we have parameterized expressions for an object shot out of a cannon with horizontal velocity 10 feet/second and vertical velocity 10 feet/second minus the acceleration due to gravity.

\[
xpos2 := (t) \rightarrow 10 \cdot t
\]

\[t \rightarrow 10 t \tag{9.13}\]

\[
ypos2 := (t) \rightarrow 10 \cdot t - \frac{32 \cdot t^2}{2}
\]

\[t \rightarrow 10t - 16 t^2 \tag{9.14}\]
Let's look an animation. First we have to create it. We can use the `animate` function of the `plots` package. For the time being, let's not worry the details of why the operation is entered the way it is. Rather we focus on what the animation is trying to do, and how to view it in Maple once it has been created.

**Table 9.6: First animation example**

This produces a movie of a point moving through the points (0,0), (.1, .01), (.2, .04), etc. up to (10, 100).

```maple
with(plots):

plot([xpos2(t), ypos2(t), t = 0 .. 5], scaling = constrained, color = "blue", labels = ["x", "y"])
```
If we click on the plot, the Maple tool bar changes and shows us *animation controls*.

**Table 9.7: Animation controls**

<table>
<thead>
<tr>
<th>Text</th>
<th>Math</th>
<th>Drawing</th>
<th>Plot</th>
<th>Animation</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="animation_controls" /></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

These controls are highly similar to video playback controls found in many applications (e.g. You Tube), so we won't discuss at length here. Note that using them you can:

1. Start playing the animation.
2. Stop the animation.
3. Display only a particular frame, "frozen".
4. Control the number of frames per second it plays.
5. Set it to play once or continually repeat in a loop.

See Graphics->Animation->Animation Toolbar under the Table of Contents of the on-line Maple help.

Right-clicking on the animation will also produce a menu of operations that provide an alternative for controlling the animation.
Table 9.8: Animation pop-up menu

<table>
<thead>
<tr>
<th>Animation pop-up menu</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Animation pop-up menu" /></td>
</tr>
</tbody>
</table>

If we play the animation, we will see a point move upwards in a parabolic path:

**Frames 1, 5, 10, 15, 20, and 25 of the animation**

<table>
<thead>
<tr>
<th>Frame</th>
<th>t = 0.0</th>
<th>t = 1.6667</th>
<th>t = 3.7500</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Frame 1" /></td>
<td><img src="image" alt="Frame 2" /></td>
<td><img src="image" alt="Frame 3" /></td>
<td></td>
</tr>
</tbody>
</table>
Frames 1, 5, 10, 15, 20, and 25 of the animation

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>Position</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.000</td>
<td>(0,0)</td>
</tr>
<tr>
<td>0.4667</td>
<td>(0.41667, 0.41667^2)</td>
</tr>
</tbody>
</table>

9.8 The first animate example, part 2

Now that we've gotten the general idea of what animate's results are like, let's look again at what was entered and look at the details of what was computed. The general form of the operation is:

<table>
<thead>
<tr>
<th>General form of the animate operation</th>
</tr>
</thead>
<tbody>
<tr>
<td>`with(plottools):</td>
</tr>
<tr>
<td><code>animate(plotting function, [ parameters to plotting function], time variable = time range)</code></td>
</tr>
</tbody>
</table>

What this does is to apply the plotting function to the parameters for various values of the time variable. Each value of the time variable used is a separate frame of the movie.

In the previous section we used:

```markdown
animate(plot, [ [t], [t^2], style = point, symbolsize = 30, color = "Purple"], t = 0..10)
```

In the first frame of the movie, it used the value \( t = 0 \) for the time variable \( t \). Thus it plotted

```markdown
plot([0],[0],style=point, symbolsize = 30, color="Purple");
```

plots a small circle at the coordinate (0,0).

The next time value used is \( t = 0.4667 \). The second frame of the movie is the result of

```markdown
plot([0.41667],[0.41667^2], style=point, symbolsize = 30, color="Purple");
```

which plots a small circle at the coordinate (.41667, 0.1736138889).
9.9 Designing a function to use in animate

In this section, we explore the design of a function that we use with animate to create an animation.

Problem

Create an animation of a circle whose radius expands from r=1 at time t=0 to r=5 at time t=10.

Solution

The actions we create are a short script that defines a function drawCircle(t). Then it uses it in animate. While we there are other ways to use animate other than this style, we choose to do things this way because the drawCircle function allows us to test the plotting one frame at a time before we try to create the animation.

Solution, part 1

```
with(plottools) :
with(plots) :

display([circle([0, 0], 1, color = "red")])
```

While this seems to do the job that we wish for the starting frame, we have to consider how this will vary as t=0 to t=10. We want the radius of the circle to be 1 when t=0 (as in the figure above), and 5 when t=10. We can do some line fitting to come up with the formula if we can't remember enough high school algebra to figure it out ourselves:

Figuring out the formula for the radius as a function of time

```
radT := CurveFitting[LeastSquares]([[0, 10], [1, 5], t])
```

```
1 + \frac{2}{5} t
```

We plot the formula we get from the data fitting with the data points to convince ourselves that the line agrees exactly with the two points. We'd
get a similar result for curve fitting a line any time we fit a line with only two data points.

Testing the solution function

\[
\text{drawCircle} := (t) \rightarrow \text{display}\left[\text{circle}\left(0, 0, 1\right) + \frac{2}{5} t, \text{color} = \text{"red"}\right]\]

\[
t \rightarrow \text{plots:-display}\left[\text{plottools:-circle}\left(0, 0, 1\right) + \frac{2}{5} t, \text{color} = \text{"red"}\right]\]

We can create a function that creates a circle of the appropriate size given a value for \( t \).
We try this function out at a $t=0$.

The function evaluated at $t=5$ looks the same, but we see from the axis labels that it is actually a bigger circle.
Once we are convinced that the function `drawCircle` works for the range of values of $t$ that we need it to, we can use it in `animate`. The way we invoke `drawCircle` within `animate` is different than when we were using it one frame at a time. The name of the function is kept separate from the parameters, and the range. This delays the creation of any plot structures until `animate` starts computing them.

**Complete script solving the "expanding circle animation" problem**

```
with(plots):
with(plottools):

drawCircle := (t) -> display
    
    circle([0, 0], 1 + \frac{2}{5}t, color = "red")

    t->plots:-display
        
        [plottools:-circle([0, 0], 1 + \frac{2}{5}t, color = "red")]
```

(9.17)
Some frames from the expanding circle animation

```latex
\texttt{animate(drawCircle, [t], t = 0..10)}
```
9.10 Designing a more elaborate animation

Problem

A satellite is in a circular orbit around the earth, at a distance of five earth radii. Create an animation of it circling.

Solution -- discussion

We will create a function drawPlanetAndSat(t) that for any time t draws both the Earth and the satellite at time t. We browse through the on-line plottools package and discover the function disk(c,r, color=....) that draws a solid disk whose center is at the point c, has radius r, and specified color, in a fashion similar to the circle function we used in the Designing a function to use in animate (page 133). Browsing through on-line help for the color names known to plot reveals that one color is "DarkKhaki". We decide to draw the Earth as a disk of radius 1 centered at (0,0).

<table>
<thead>
<tr>
<th>Drawing a disk</th>
</tr>
</thead>
<tbody>
<tr>
<td>with(plots) :</td>
</tr>
<tr>
<td>with(plottools) :</td>
</tr>
<tr>
<td>display([disk([0,0], 1, color = &quot;DarkKhaki&quot;)])</td>
</tr>
</tbody>
</table>

We decide to represent the satellite as a point. Recall that (cos(t), sin(t)) describes circular motion moving around a circle of radius r=1. To parameterize orbital motion at radius 5, we use (5*cos(t), 5*sin(t)).

<table>
<thead>
<tr>
<th>Drawing a point at (0,5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>with(plots) :</td>
</tr>
<tr>
<td>with(plottools) :</td>
</tr>
</tbody>
</table>
We can combine the two together using `display`.

```plaintext
plot(5*cos(0), 5*sin(0), style = point, symbolsize = 20, color = "red")
```

We invoke `display` with a list of two plot structures: the disk, and the point plot.

After we draw this we see the two plots together, but it isn't exactly what we want, because the horizontal and vertical scaling is not equal.

**Drawing a disk, and a point at (0.5)**

```plaintext
with(plots):

with(plottools):

display([disk([0, 0], 1, color = "DarkKhaki"),
    plot([5*cos(0), 5*sin(0)], style = point,
        symbolsize = 20, color = "red")])
```
We include the plot option `scaling=constrained` as an extra argument to `display`.

We're almost done. As with the animation design of the previous section, we write a function of \( t \) that describes the disk and the position of the point at time \( t \).

**A function describing a frame to draw at time \( t \)**

```maple
with(plots):

with(plottools):

drawEarthAndSat := (t) -> display([disk([0, 0], 1, color = "DarkKhaki"), plot([5*cos(t), 5*sin(t)], style = point, symbolsize = 20, color = "red"), scaling = constrained])

t -> plots:-display([plottools:-disk([0, 0], 1, color = "DarkKhaki"), plot([5*cos(t), 5*sin(t)], style = plottools:-point, symbolsize = 20, color = "red"), scaling = constrained])
```

(9.18)
We test this function out at a few values of $t$. The idea is that the satellite makes one orbit during the period $t=0..2\pi$.
We now can use this function with `animate` to draw the orbiting satellite.

**Complete script to solve the "orbiting satellite animation problem"**

```plaintext
with(plots):

with(plottools):

drawEarthAndSat := (t) -> display([disk([0, 0], 1, color = "DarkKhaki"), plot([5*cos(t), 5*sin(t)], style = point, symbolsize = 20, color = "red");}, scaling = constrained)

t -> plots::display([plottools::disk([0, 0], 1, color = "DarkKhaki"), plot([5*cos(t), 5*sin(t)], style = plottools::point, symbolsize = 20, color = "red");}, scaling = constrained)
```

(9.19)
We have the animation have the satellite circling three times. Some of the frames of the animation can be seen in the table of frames from the orbiting satellite animation (page 143).

### Table of selected frames from the orbiting satellite animation

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>Frame 1</th>
<th>Frame 2</th>
<th>Frame 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td><img src="image1" alt="Frame 1" /></td>
<td><img src="image2" alt="Frame 2" /></td>
<td><img src="image3" alt="Frame 3" /></td>
</tr>
<tr>
<td>5.333</td>
<td><img src="image2" alt="Frame 2" /></td>
<td><img src="image3" alt="Frame 3" /></td>
<td></td>
</tr>
<tr>
<td>28.744</td>
<td><img src="image3" alt="Frame 3" /></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### 9.11 Another animation example

#### Problem

A boy throws a ball straight up in the air with an initial velocity of 15 miles per hour. Once released, the ball's position is described by the function $x(t) = vt - \frac{1}{2}gt^2$, where $v$ is the initial velocity, and $g$ is the force of gravity, $g = \frac{32 \text{ feet}}{\text{sec}^2}$.

(a) How many seconds does the ball stay in the air?

(b) Generate an animation that shows the ball's motion. $t$ should be measured in seconds, and position in feet from ground level. Assume that the ball starts at 0 feet altitude even though this would be a bit unrealistic for a boy to do unless he was standing in a pit!

(c) Use the animation to determine roughly when the maximum altitude is, and what that altitude is.
(d) Find how fast the initial velocity should be so that the ball goes up over 30 feet.

**Solution**

First convert 15 miles per hour into a velocity in feet per second using the `convert` function first described in the *section on problem solving functions in Chapter 7* (page 102).

\[ v_0 := \text{convert} \left( 15, \text{units}, \frac{\text{miles}}{\text{hour}}, \frac{\text{feet}}{\text{second}} \right) \]

Next, define the gravitational constant

\[ g := -32 \]

Next, define the function for position.

\[ x := (t) \rightarrow v_0 \cdot t + \frac{g \cdot t^2}{2} \]

Doing a rough plot of position versus time shows us roughly when the ball will hit the ground. We guess that it might take three seconds.

\[ \text{plot}(x(t), t = 0 \ldots 3) \]

Oh, that's too much. The answer seems to be about 1.4 seconds, but we can use `solve` to come up with an exact value.

\[ \text{solve}(0 = x(t), t) \]

\[ 0, \frac{11}{8} \]

There are actually two solutions -- the obvious one is when \( t=0 \) and the ball hasn't yet been thrown.
To get the larger one, we compose the \textit{max} function with \textit{solve}.

\[
\text{flightTime} := \max(\text{solve}(x(t) = 0, t))
\]

\[
\frac{11}{8}
\]

(9.24)

This answers part (a) of the problem.

We verify our computation by evaluating \(x\) at that time and finding that the position of the ball really is at altitude 0.

\[
x(\text{flightTime})
\]

\[
0
\]

(9.25)

Now we need to make the movie. We need to create a function which for time \(t\), draws a point at the coordinate \((0, x(t))\). To illustrate what we mean, let's compute the position of the ball at \(t=1\) second.

\[
x(1.0)
\]

\[
6.000000000
\]

(9.26)

We want the ball to be at position \((0,6)\). A plot command that would do this would be:

\[
\text{plot([0], [6], style = point, color = red, symbol = circle, symbolsize = 30)}
\]
Similarly, at $t=.5$, the ball's position would be at $x(0.5)$:

$$\text{plot}([0], [x(0.5)], \text{style = point, color = red, symbol = circle, symbolsize = 30})$$

Evidently $x(.5)$ is 7.

We create a user-defined function that creates a plot structure as a result.

$$\text{ballFrame} := (t) \rightarrow \text{plot}([0], [x(t)], \text{style = point, color = red, symbol = circle, symbolsize = 30})$$

$$t \rightarrow \text{plot}([0], [x(t)], \text{style = point, color = red, symbol = circle, symbolsize = 30})$$  \hspace{1cm} (9.27)$$
Let's try this out for $t=.5$ and see if we get the same result.

\[ \text{ballFrame}(0.5) \]

We can now use this function with \textit{animate}. Note that we needed \textit{flightTime} in order to describe how long the movie runs. If we did more or less than that, then the ball wouldn't have landed, or would be shown as going below ground level.

\begin{verbatim}
with(plots):

animate(ballFrame, [t], t = 0 .. flightTime)
\end{verbatim}

This answers part (b) of the problem.
By playing the movie, we see that the maximum altitude is about 7.5 feet, at time t=0.687 seconds.

This answers part (c) of the problem.

To answer part (d), we need to do more programming. We first modify the plot so that it draws a line segment at 30 feet as well as plotting the position of the ball. We use the `line` function of the `plottools` package discussed in an ?? to draw a line segment at (-5,30) to (5,30), and to color it green:

\[
\text{height} := 30
\]

\[
\text{with(plottools)}:
\]

\[
pLine := \text{line}([-5, \text{height}], [5, \text{height}], \text{color} = "\text{green}")
\]

\[
\text{CURVES([[} -5.,30.\text{],[}5.,30.\text{]], COLOUR(RGB, 0., 1.0000000, 0.))}
\]

The display function can be used to combine this line with a frame of the movie. Here is an example of this:

\[
\text{with(plots)}:
\]

\[
\text{display([ballFrame(0.5), pLine])}
\]

The automatic scaling of plot chops off the vertical distance between 6 feet and 0 because there is nothing in this frame that needs that. In the animation, the scale is adjusted so that all frames operate in the same axes.

Now we can create a new user-defined function that plots both the line and the ball.

\[
\text{ballWithLine} := (t) \rightarrow \text{display([ballFrame(t), pLine])}
\]

\[
t\rightarrow \text{plots\-display([ballFrame(t), pLine])}
\]
To look at the behavior of the ball at a particular velocity, we can now execute a two line script, consisting of assigning \( v_0 \) to the desired initial velocity, and then the operation that draws the movie.

\[
v_0 := \text{convert} \left( 15, \text{units}, \frac{\text{miles}}{\text{hour}}, \frac{\text{feet}}{\text{second}} \right)
\]

\[22\]

\( \text{animate} \left( \text{ballWithLine}, [t], t = 0 .. \text{max}(\text{solve}(x(t) = 0, t)) \right) \)

As we already have seen, \( v_0 = 22 \) is not fast enough. We set it higher and recalculate the movie:

\[
v_0 := 50
\]

\[50\]

\( \text{animate} \left( \text{ballWithLine}, [t], t = 0 .. \text{max}(\text{solve}(x(t) = 0, t)) \right) \)
That was too high. Let's try 40.

\[ v \theta := 40 \]  

\[ \text{animate}(\text{ballWithLine}, \{t\}, t = 0..\max(\text{solve}(x(t) = 0, t))) \]

Too low. Let's try 45

\[ v \theta := 45 \]  

\[ \text{animate}(\text{ballWithLine}, \{t\}, t = 0..\max(\text{solve}(x(t) = 0, t))) \]

So 45 feet per second seems to be about right. We could get a more precise determination through movie-watching, but for high accuracy we should use more mathematics. In a subsequent chapter, we will introduce additional Maple operations that can calculate the velocity exactly (or a close approximation) without the trial-and-error of movie watching. Having the movies did give us a better understanding of the phenomenon.
9.12 Exporting animations and non-animated plots

One operation available in the popup menu is Export. Right-click (or control-click) -> Export -> Graphics Interchange Format will produce an animation file in .gif format. As the animation file is being created, a dialog box will appear asking you to specify the directory where the .gif file should be written. Once created, the file can be included on web pages or other documents.

This feature is also available for ordinary (non-animated) plots. Right-clicking (control-click for Macintosh) will create a file of the plot in .gif, .jpeg, or .ps format. However, .gif file is the only format of the three that is supported by web browsers for animations.

Table 9.9: Exporting animations through the pop-up menu
### 9.13 Summary of Chapter 9

**Combining plots and shapes using display**

Supplying the `display` function with a list of two or more plots will cause those plots to be plotted on top of one another.

We can save a plot in a variable to display later. Marking a specific value on the plot can be accomplished using the `line` or `circle` function.

```
plot4 := plot(sin(x), x = 0 .. 2*Pi)

maxAmp := line([0, 1], [2*Pi, 1], color = 'blue')
CURVES([[0, 1], [6.283185308, 1]], COLOUR(RGB, 0, 0, 1.0000000))
```

```
figure

display([plot4, maxAmp])
```

**Parameterized plots**

For plots that may have values corresponding to 2d positions, we use multiple functions to define both \( x(t) \) and \( y(t) \).

\[
xpos := (t) \rightarrow 3 \cdot \cos(t)
\]

\[
ypos := (t) \rightarrow 2 \cdot \sin(t)
\]
### Parameterized plots

We put these functions in a list, along with the range of the independent variable \( t \), and plot.

\[
\text{plot}([x(t), y(t), t = 0..\pi], \text{scaling} = \text{constrained}) : 
\]

### Animating plots using animate

#### Creating an animation

\[
x := (t) \rightarrow 50 \cdot \sin\left(\frac{\pi}{4}\right) t : \\
y := t \rightarrow 50 \cdot \cos\left(\frac{\pi}{4}\right) t - 16 t^2 : \\
posPlot := (t) \rightarrow \text{plot}([x(t)], [y(t)], \text{style} = \text{point}, \text{symbol} = \text{circle}) : \\
\text{animate}(posPlot, [t], t = 0..2) : 
\]

#### Controlling the animation using the animation tool bar or the animation popup menu

The animation tool bar is located above the workspace window, while the popup menu can be displayed by right-clicking the animation.

#### Exporting an animation to a graphics file

Right click animation > Export > Graphics Interchange Format
10 Moving into programming

10.1 Chapter Overview

The interactive document interface that we've worked with so far is good for quick development of calculations involving a few steps. Because feedback occurs after each step entered, this style of working is often the fastest way of getting such short calculations done. We've also seen how the effort to develop a script can be made to yield greater payback by finding and exploiting situations that require re-use, by editing parameters of a script and re-executing it.

In our previous work, we've executed a script by the following process:

a) Setting up the values needed for the parameters assigned at the beginning of the script. This makes it easy to find and change the parameter values when you want to re-use it.

b) Listing the actions to solve the problem after the parameters are listed. The notion is that if all you want is re-use, you don't change anything in this part of the script.

b) Positioning the mouse cursor on the first line of the script and hitting return (or enter) repeatedly, hitting the !!! button on the Maple Toolbar, or selecting Edit->Execute->Selection.

With longer scripts or extensive re-execution, this process becomes tedious even if we are saving a lot of time by having the computer do the work. In this chapter, we introduce a new way of entering and executing scripts: code edit regions. While such regions make it more convenient to work and execute blocks of Maple instructions, the programmer must enter things using the textual version of Maple expressions and operations. Individual lines must be separated by semi-colons (;) so that the Maple language processor can more easily tell where one instruction ends and the next begins.

With code edit regions, the programmer must also work harder to see how the state of the computation is changing during the computation, rather than only looking at the final output. This is because if the final result is wrong, the programmer must find at what point in the script mistakes (called program bugs) occur. Work with code edit regions often requires adding print or printf statements into the script to better see changes in variables and intermediate results.

10.2 More on printing: print, printf and sprintf

print is a function that takes a sequence of values as arguments, and returns NULL (first introduced in ???) as a result. Its use is in its side effect -- it causes the sequence of values to be displayed in the worksheet in two dimensional format. print is used to provide more intelligible displays of information by providing words to go along with quantities or formulas that are computed. The words, of course, are provided by including strings as part of the sequence.

**Example**

<table>
<thead>
<tr>
<th>Use of print</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L := \left[ \exp \left( -\frac{1}{10.0} \right), \exp \left( -\frac{2}{10.0} \right), \exp \left( -\frac{3}{10.0} \right) \right] )</td>
<td>We set up a list of values. L has all floating-point values because the &quot;10.0&quot; in the denominator causes all computations to be done using limited-precision arithmetic. The display of the value of L is suppressed because we ended the assignment with a colon.</td>
</tr>
<tr>
<td>print(&quot;This list presents some values of the function &quot;, \exp(x)); \quad \text{printf} \left( \text{&quot;. The last element of the list is:&quot;}, L[-1]) \right);</td>
<td>print prints out the sequence of values given to it as arguments. You can give any number of items in the sequence. The commas separating the string and the number are from the sequence. Note that ?? the result is not displayed because the line ends with a colon.</td>
</tr>
</tbody>
</table>

"This list presents some values of the function ", \( e^x \)

". The last element of the list is:" , 0.7408182207

(10.1)
printf is a function that returns NULL (first introduced in ???) as its result, and is used primarily for its side effect of causing information to be displayed in the worksheet when the code is executed. The first item in the sequence is a string that contains ordinary words plus special format codes that begin with %. ???summarizes the commonly used the format codes. The on-line documentation for printf describes all format codes comprehensively. Some of them allow for quite intricate effects such as right or left justification, padding with leading or trailing zeroes, etc.

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>printf(&quot;There are &quot;, nops(L), &quot; elements in this list. The largest value is&quot;, max(L))</code></td>
<td></td>
</tr>
<tr>
<td>&quot;There are &quot;, 3, &quot; elements in this list. The largest value is&quot;, 0.9048374180</td>
<td>(10.2)</td>
</tr>
</tbody>
</table>

Use of printf

We set up a list of values as in the previous example.

| `L := [exp(-.1), exp(-.2), exp(-.3), 4/5];` | |
| `printf("The last element of the list is \%f\n", L[-1])` | |
| The last element of the list is 0.800000. | |

The first argument to printf is a string describing how to format the rest of the values. The string contains format codes describing where to insert the values of the later arguments into the message described in the string. In this example, there is one format code \%f, for printing numbers in conventional decimal point notation. Since printf joins together the message and the value together into a single message without a sequence into a single result, what is printed is without commas.

Additional printf codes are "\%d" (for integer values), and "\%e" (for floating point numbers in scientific notation). The format code 'n causes the next output to occur on the next line.

There are 4 elements in this list. The largest is 9.048374e-01.

L has 4 elements. The largest is 9.048374e-01.

If you attempt to use \%f format on an integer, or \%d format on a number that is not an integer, you will get incorrect values (with no warning that they are incorrect), or possibly an error message. Let the user beware; the computer isn't going to necessary save you from making a mistake.

You can't print a list of floating point numbers using just \%f format.

The \%a format will handle arbitrary numerical or non-numeric values. The value being printed out is a list, which is a "non-numeric value" because it isn't a single number. However, the output uses the textual format rather than the "pretty" format used by printf.
sprintf is like printf in that its first argument is a string with format codes, and the rest of its arguments are values to be formatted. However, unlike printf it does not print (display) any information. It returns a string which is the formatted information instead. This is useful in situations where functions need formatted information, such as titles or labels of plots.

Table 10.1: Use of sprintf

<table>
<thead>
<tr>
<th>Code</th>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>L := [exp(-1), exp(-2), exp(-1.5), exp(-2.7)] : message := sprintf(&quot;plot of x versus exp(-x), using %d points&quot;, nops(L))</td>
<td>We set up a list of data points. Using sprintf, we create a string. The %d format is used to insert the number of points into the message.</td>
<td></td>
</tr>
<tr>
<td>plot([-1., -2., -1.5, -2.7], L, style = point, title = message, symbolsize = 30)</td>
<td>We need the string to put a title onto our pointplot. You can read more about the title= argument to pointplot by entering plot,options in Maple's on-line help. The advantage of using sprintf is in re-use. If we want to plot a different set of points, we can change the line assigning L (and the plot line of x coordinates), but we don't have to alter the sprintf since it will automatically use the right value corresponding to the length of the list.</td>
<td></td>
</tr>
</tbody>
</table>

Table 10.2: Common formatting codes for printf and sprintf

<table>
<thead>
<tr>
<th>Format code</th>
<th>Example</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>%a</td>
<td>printf(&quot;The answer is: %a&quot;, x = x^2 + 1)</td>
<td>Algebraic format. Works for any type of value. The value is printed out in a format suitable for textual input.</td>
</tr>
<tr>
<td>%f</td>
<td>printf(&quot;The answer is : %f&quot;, L[-1])</td>
<td>Fixed format for floating point numbers. w and d are optional. If given, w describes the number of columns for the entire number, and d the number of columns for the number after the decimal point.</td>
</tr>
<tr>
<td>%w.df</td>
<td>printf(&quot;The answer is : %w.df&quot;, L[-1])</td>
<td></td>
</tr>
</tbody>
</table>
### Math Functions

<table>
<thead>
<tr>
<th>Format code</th>
<th>Example</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>printf(&quot;%10.5f&quot;, L[-1])</code></td>
<td>0.00910</td>
<td>Values will be converted to floats where possible.</td>
</tr>
<tr>
<td><code>printf(&quot;%10.5f&quot;, 20!)</code></td>
<td>2432902008176640000.00000</td>
<td></td>
</tr>
<tr>
<td><code>printf(&quot;%16.3f&quot;, 3/4)</code></td>
<td>0.750000</td>
<td></td>
</tr>
<tr>
<td><code>printf(&quot;The answer is: %15.6f, x=x^2 + 1&quot;)</code></td>
<td>Error, (in fprintf) number expected for floating point format</td>
<td></td>
</tr>
<tr>
<td><code>%e</code></td>
<td><code>printf(&quot;%12.5e&quot;, L[-1])</code></td>
<td>9.095277e-03</td>
</tr>
<tr>
<td><code>%w.de</code></td>
<td><code>printf(&quot;%12.5e&quot;, 20!)</code></td>
<td>9.09528e-03</td>
</tr>
<tr>
<td></td>
<td><code>printf(&quot;%10.5e&quot;, 10!)</code></td>
<td>3.628800e+06</td>
</tr>
<tr>
<td></td>
<td><code>printf(&quot;%16.3f&quot;, 3/4)</code></td>
<td>7.500000e-01</td>
</tr>
<tr>
<td></td>
<td><code>printf(&quot;The answer is: %15.6e, x=x^2 + 1&quot;)</code></td>
<td>Error, (in fprintf) number expected for floating point format</td>
</tr>
<tr>
<td><code>%d</code></td>
<td><code>printf(&quot;%d&quot;, 20!)</code></td>
<td>2432902008176640000</td>
</tr>
<tr>
<td></td>
<td><code>printf(&quot;%16.3f&quot;, 3/4)</code></td>
<td>Error, (in fprintf) integer expected for integer format</td>
</tr>
<tr>
<td></td>
<td><code>printf(&quot;%d&quot;, 3.0)</code></td>
<td>Error, (in fprintf) integer expected for integer format</td>
</tr>
</tbody>
</table>

- **%e** format code: Scientific format for floating point numbers. `w` and `d` are optional. If given, `w` describes the number of columns for the entire number, and `d` the number of columns for the number after the decimal point. If `w` is wider than the number of digits required, then it is filled out with leading white space.
- **%w.de** format code: Values will be converted to floats where possible.
- **%d** format code: Format for integers. This format rejects any value that is not of integer type. The built-in `whattype` function can provide guidance about what is and what isn't going to work, if you aren't sure.
<table>
<thead>
<tr>
<th>Format code</th>
<th>Example</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>%f</td>
<td><code>float</code></td>
<td>(10.7)</td>
</tr>
<tr>
<td>%d</td>
<td><code>integer</code></td>
<td>(10.8)</td>
</tr>
<tr>
<td>%s</td>
<td>Format for strings.</td>
<td></td>
</tr>
<tr>
<td>%n</td>
<td>Causes a line break between preceding and succeeding parts of the output.</td>
<td></td>
</tr>
<tr>
<td>%t</td>
<td>Causes next output to appear starting at the next columnar (tab) position.</td>
<td></td>
</tr>
</tbody>
</table>

### 10.3 For the curious: why printf is so cryptic

The format codes for `printf` are a "programming language within a programming language" in that they describe in abbreviated form how the computer should output values. The style used for format codes is very much unlike the rest of Maple, in that brevity rather than readability seems to be the criterion used to design this portion of the language. A language for science and engineering popular starting in the middle of the twentieth century was Fortran, which had formatting that was similar in approach but was much different in syntax. Given your knowledge of Maple (and English), the intent of the program is fairly obvious: it will print There were 3 values, and the answer was 4.7.

A Fortran program that demonstrates formatted printing

```fortran
n = 3
x = 4.7
print (*,35) n,x
35 format('There were',i2,' values, and the answer was',f5.1, '.')
stop
end
```

The design of `printf` can be explained by the fact that Maple's `printf` is a close imitation of the `printf` found in the programming language C (popular with many programmers at the time Maple was invented in the '80s, and still popular among computer engineers and scientists doing device programming). Matlab, being invented in the same era, also a `sprintf` function of highly similar design. Imitation makes it easier to transfer programming skill from one language to another. Inventors of programming languages will tend to imitate unless they have a goal or insight that justifies a difference. Programming languages designed by skillful designers...
do not introduce arbitrary differences without a good cause. When you notice similarities between languages, take it as a recognition that many people liked that style of doing things, across many different kinds of programmers and kinds of problems. When you notice a difference between languages, you should explore the reasons why the designers made different choices. Usually there is a problem or issue that the designer is trying to handle better by being different from the rest of the pack.

Programming language features features that work across tens of thousands of programmers and programs are usually motivated by significant design rationales. The design of most programming languages is nowadays informed by what worked well or didn't work well in the past.

10.4 Code edit regions: executing a series of actions at once

We have seen that even some of the easiest technical calculations break down into a series of operations, chained together. Realizing that we will typically need the computer to perform a "series of operations" is the motivation for making the transition to programming, where execution of many operations in a block is typical.

The Maple document interface that we have been using so far easily supports calculations where the user prompts to computer to do a series of steps by positioning the cursor at the first operation and then hitting return (or enter) repeatedly. We now introduce a second way to enter instructions and have them executed as a block. This alternative is often easier to work with when you have a few dozen instructions and will want to execute them all in a chain.

One can open a text field in the Maple document where one can enter a series of instructions. To do this, position your cursor where you want the text field to appear in the document, and with the mouse perform Insert->Code Edit Region

Insert->Code Edit Region menu
Once the field has been created, you can type the textual version of the Maple instructions you want executed. Each instruction (commonly referred to as a *statement*) must be separated by either a semi-colon, or a colon. If the statement ends in a semi-colon, then its value will be printed during execution of the region. If the statement ends in a colon, then printing of its value will be suppressed just as it is in operation of documents.

Once all the instructions have been entered, you can run them all in a series by typing control-e (command-e on Macintosh), or by entering right-click->Execute Region (on Macintosh, control-click->Execute Region). Instead of typing control-e, you can as an alternative click on the "!!" icon on the Maple toolbar. The results for each statement will appear below the region in blue, except for the statements whose printing is suppressed by a colon.

**Execute Group icon ("!!" icon) on Maple Toolbar**
Any portion of a line that begins with a "#" is a program comment. The rest of the line is regarded by Maple as something that people will read, not an instruction to be performed by the computer. Typically what appears after the # is commentary written in English (or whatever language is convenient for communication with the intended audience) that helps explain/remind human readers what a segment of code is about. As you read programs with comments, you will see that some programmers are "chattier" than others. This is because they feel that their intended reading audience needs more explanation.

In professionally written code, you will often see comments at the beginning of the region that give an overview of what the code region does, the name of the author(s), and the date of creation/modification. Professional organizations often mandate a particular style and content for such "header" comments, so that it will be easy to find information about any code written by several programmers working on a single project.

### A code edit region with actions entered, separated by semi-colons.

<table>
<thead>
<tr>
<th>Code region</th>
<th>Commentary</th>
</tr>
</thead>
</table>
| # Figure 11.1.1  
#initialize variables  
i := 1;  
val := .3; #evaluation point  
term := (val^i)/i!; # a term to compute  
print("term is", term); #message  
s := term; #s has a copy of the term  
tol := 10e-7; #A small value. | Segments of lines that begin with # are regarded as program comments (for the program reader's eyes), not operations for Maple to carry out.  
The results in blue are displayed after we position the cursor in the code edit region and type control-E (command-E on Macintosh), or enter Execute Code Region via the clickable menu.  
Result of first assignment to \( i \)  
Result of assignment to \( val \).  
Result of assignment to \( \text{term} \).  
Result of calling the \( \text{print} \) function.  
Result of assignment to \( s \).  
Result of assignment to \( \text{tol} \). |

| 1 |
| 0.3 |
| 0.3 |
| "term is", 0.3 |
| 0.3 |
| 0.0000010 |

The code region will develop a scroll bar if the amount of text entered exceeds the size of the window. The size of the field can be adjusted by right click->Component Properties, and then modifying the integers listed for the width and height.

**Enlarging a code Region window by changing component properties via the clickable menu**
# Figure 11.1.1
#initialize variables
i := 1;
val := .3; #evaluation point
term := (val^i)/i!;  # a term to compute
print("term is", term); #message
s := term;  #s has a copy of the term
tol := 10e-7;  #A small value.
The "Collapse Code Edit Region" menu item of the clickable menu will reduce the entire window to an icon. If the first line of the region is a program comment, it will be listed to the right of the icon. Clicking on the code icon will execute the code within.

A collapsed code edit region
10.5 Turning intentions into code

More complicated problems or longer computations requires a planning phase before any programming is done. Once the plan has been developed, the programmer then goes about the business of writing the actions described in the plan in the programming language being used. This business of translating intentions into instructions in a language a computer can understand is called coding, or writing program code. In concrete terms for this course, what this means is that the problems we will be working on will need some planning before anything is written in Maple. As with all plans, it is beneficial to be able to see how well the plan is going to work (i.e. to evaluate the plan) and fix/polish it before moving to the coding phase.

The purpose of programming languages is to express a solution to the computer. They are not ideal languages for people to think in to develop the outline of a solution.

10.6 Making code easier to read and understand

As has been mentioned ???, an important part of training for a programmer is to make code easier to reuse, since reuse allows the cost of writing the program to be amortized over more invocations of it. Part of reuse is allowing others to understand what they should do with the code, or to allow yourself to more quickly recall the details of what you were doing when you return to a program after a few weeks or months away from it. This kind of reuse is helped by making certain s tylistic choices in how you write the code. Here are some ideas:

1. Use mnemonic variable names -- names that suggest what the variable's purpose is. This is different from the variable naming practices of the "mathematics culture", which is to use symbols of one letter in length, e.g. $x$, $i$, $\alpha$.

   The reason for the difference in practices is that it is relatively easy to keep track of the purpose of variables if the expression is only one line long, as is typical in elementary mathematics. Even elementary programs however consist of many lines. A first-year student's program might be a dozen or even 100 lines, using ten or twenty variables. Remembering what they all do becomes too hard without reminders.

2. Invent and use user-defined functions to encapsulate common operations. Reusing a name two or more times has advantages beyond not doing so much typing. It allows the reader to see quickly that you are doing the "same thing as before" instead of something subtly or radically different.

3. Mnemonically name functions. If the function has a name that describes what is doing (e.g. "plotCoolingFunction" instead of "f") then it becomes even easier for the reader to understand/recall what the function is intended to do even the first time that they read about it.

4. Blank lines are usually used to indicate the natural conceptual divisions between different groups of instructions. For example, the first few instructions establishing initial values of variables might be separated by a few blank lines from the rest of the instructions which establish the main body of work, which in turn may be separated by the "finishing up" instructions.

5. Indentation is used for indicating nuances of control, such as the extent of code blocks that are repeated. We will discuss indentation more in the next chapter where the kind of code that benefits from it is first discussed.
10.7 Troubleshooting common errors in entry

New kinds of errors and warnings can appear in code regions, due primarily to the new requirement that statements be separated by semicolons and colons. Trying to enter too many lines at once before testing what you have so far can make older errors appear more mysterious.

Table 10.3: Example Commentary

<table>
<thead>
<tr>
<th>Code regions with entry errors</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Example</strong></td>
<td><strong>Commentary</strong></td>
</tr>
</tbody>
</table>
| \[ a := \sin(3.5 + \cos(\frac{\pi}{2}) ; \]
| Error, `;' unexpected         | The error message indicates that a semi-colon was not expected where Maple scanned it. This is typically semi-colons can appear only at the end of a complete Maple statement. This message typically indicates that something went wrong in what was entered before the semi-colon. Edit the code region to make the statement correct, and re-execute the region. |
| (Above region with fixes)     | The problem here is a missing parentheses ). To fix this, you'd put a closing ) after the "3.5", and then re-execute the region with a control-e(or command-e). |
| \[ a := \sin(3.5) + \cos(\frac{\pi}{2}) \] | This is only a warning, and executing this region does perform the computation correctly. |
| Warning, inserted missing semicolon at end of statement | |
| \[ a := \sin(3.5) + \cos(\frac{\pi}{2}) ; b := \cos(3.5) + \sin(\frac{\pi}{2}) \] | If we add another line without a semi-colon between, we get just an error. Both lines are messed up because Maple needs an explicit separator between the statements. Just putting the next instruction on another line is not enough. |
| Error, missing operator or `;' | |
| (Above region edited with fixes) | If you try to enter several lines at once before entering, you have the additional problem of figuring out which line the mistake occurred on. In this case, you have to decide which of the two semi-colons was unexpected. |
| \[ a := \sin(3.5) + \cos(\frac{\pi}{2}) ; b := \cos(3.5) + \sin(\frac{\pi}{2}) ; \] | There's both a missing closing parentheses and a missing semi-colon here. The symptom is that there is a warning message (which isn't necessarily a sign of trouble, but is here), plus the fact that you are expecting a value to be printed from the computation but just see a red ">", with no numeric output. Fixing both problems and re-executing the region will produce the correct result. |
| Error, `;' unexpected         | Sometimes when you enter a line, you make more than one error. Even though the same mistake has been made as in the previous example, |
| \[ a := \sin(3.5 + \cos(\frac{\pi}{2}) \] | |
| Warning, premature end of input, use <Shift> + <Enter> to avoid this message. | |
there is a different error message than before. Executing this region causes the message indicated, plus the cursor will be observed to flash at the start of the second line.

Rather than being in an "I was expecting more" state which produced the first warning message, Maple is in a "I was expecting more, but wait what I've seen isn't what I was expecting". Since there's nothing before the cursor started flashing before any of the second line was parsed, it means that the error must have happened before that on the previous line.

There are still two things to fix. Fixing them will produce two lines of output.

**10.8 Summary of Chapter 10**

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Use of print</strong></td>
<td></td>
</tr>
<tr>
<td>( L := \left[ \exp\left(-\frac{1}{10.0}\right), \exp\left(-\frac{2}{10.0}\right), \exp\left(-\frac{3}{10.0}\right) \right] )</td>
<td>We set up a list of values. ( L ) has all floating-point values because the &quot;10.0&quot; in the denominator causes all computations to be done using limited-precision arithmetic. The display of the value of ( L ) is suppressed because we ended the assignment with a colon.</td>
</tr>
<tr>
<td>( \text{print(}&quot;\text{This list presents some values of the function } e^x,\text{ the last element of the list is:}, L[-1]\text{);})</td>
<td></td>
</tr>
<tr>
<td>&quot;This list presents some values of the function ( e^x ), ( \exp(x) ), \text{ the last element of the list is:}, 0.7408182207)</td>
<td></td>
</tr>
<tr>
<td>( \text{print(}&quot;\text{There are } n\text{ elements in this list. The largest value is}, \text{ max}(L)))</td>
<td></td>
</tr>
<tr>
<td>&quot;There are ( n ), \text{ elements in this list. The largest value is}, , 0.9048374180)</td>
<td></td>
</tr>
</tbody>
</table>

**Example**

**Use of printf**

\( L := \left[ \exp(-.1), \exp(-.2), \exp(-.3), \frac{4}{5} \right] : \)

\( \text{printf(}"\text{The last element of the list is } \%f,\text{ the list is:}, L[-1]\)\)

The last element of the list is 0.800000.

\( \text{printf(}"\text{There are } \%d\text{ elements in this list. The largest is } \%e,\text{ nops}(L), \text{ max}(L)\)\)

There are 4 elements in this list.
The largest is 9.048374e-01.

Additional printf codes are "\%d" (for integer values), and "\%e" (for floating point numbers in scientific notation). The format code \( \text{\textbackslash n} \) causes the next output to occur on the next line.
Commentary

Example

```c
printf("L has %d elements.", nops(L));
printf("The largest is %f", max(L));
```

L has 4 elements. The largest is 9.048374e-01.

Note that a printf statement does not automatically cause the output on a new line. Thus, several printf statements executed together as these are will have all the output on a single line.

```c
printf("There are %d elements in this list. The largest is %f",
       nops(L), max(L));
```

There are 4.000000 elements in this list. The largest is

Error, (in fprintf) integer expected for integer format

If you attempt to use %f format on an integer, or %d format on a number that is not an integer, you will get incorrect values (with no warning that they are incorrect), or possibly an error message. Let the user beware; the computer isn't going to necessary save you from making a mistake.

```c
printf("%d", L);
```

Error, (in fprintf) number expected for floating point format

You can't print a list of floating point numbers using just %f format.

```c
printf("%a", L);
```

The %a format will handle arbitrary numerical or non-numeric values. The value being printed out is a list, which is a "non-numeric value" because it isn't a single number. However, the output uses the textual format rather than the "pretty" format used by print.

Table 10.4: Use of sprintf

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>L := [exp(-.1), exp(-2), exp(-1.5), exp(-2.7)];</code></td>
<td>We set up a list of data points. Using sprintf, we create a string. The %d format is used to insert the number of points into the message.</td>
</tr>
<tr>
<td><code>message := sprintf(&quot;plot of x versus exp(-x), using %d points&quot;, nops(L));</code></td>
<td></td>
</tr>
<tr>
<td>&quot;plot of x versus exp(-x), using 4 points&quot;</td>
<td>(10.16)</td>
</tr>
</tbody>
</table>

```c
Table10.4:Useofsprintf
```

Commentary

We set up a list of data points. Using sprintf, we create a string. The %d format is used to insert the number of points into the message.
Commentary

We need the string to put a title onto our pointplot. You can read more about the `title=` argument to `pointplot` by entering `plot,options` in Maple's on-line help.

\[
\text{plot([ \{-1, -2, -1.5, -2.7\}, L, style = point, title = message\})}
\]

plot of \( x \) versus \( \exp(-x) \), using 4 points

-2.5

-2

-1.5

-1

-0.5

-0.1

0.1

0.2

0.3

0.4

0.5

0.6

0.7

0.8

0.9

Code edit regions

Insert -> Code Edit Region to create a region.

Control-e (command-e in Macintosh) or

right-click -> Execute Region in a code edit region to execute all the code in it.

right-click->Component Properties to open a dialog box with the dimensions of the code region (in pixels). Change these dimensions by entering different numbers to enlarge or shrink the box. Sorry, dragging will not resize the code region.

right-click->Collapse Region to turn the code region into an icon.

Clicking on the icon will then execute the code inside.

The first line of code appears as the label for the icon.

right-click->Expand Region to turn a code region icon to display the region again.

Semi-colons and colons

All Maple statements must be separated or terminated with either a semi-colon (;) or a colon (:). Usually each statement executed generates a line of output which appears in blue after the region. This sequence of output is called the execution trace of the code. If a colon is used, then the output that normally results from evaluating the statement is suppressed from the execution trace.

Comments

Any portion of a line of code starting from a number sign (#) is treated as commentary and not as an operation to be performed.

Example

```maple
# Figure 11.1.1
#initialize variables
```

Segments of lines that begin with # are regarded as program comments (for the program reader's eyes), not operations for Maple to carry out.
The results in blue are displayed after we position the cursor in the code edit region and type control-E (command-E on Macintosh), or enter Execute Code Region via the clickable menu.

- **Result of first assignment (to \( i \))**
- **Result of assignment to \( val \).**
- **Result of assignment to \( term \).**
- **Result of calling the \texttt{print} function.**
- **Result of assignment to \( s \).**
- **Result of assignment to \( tol \).**

\[
\begin{array}{c}
1 \\
0.3 \\
0.3 \\
"term is", 0.3 \\
0.3 \\
0.0000010 \\
\end{array}
\]

(10.17)
11 Creating and developing code

11.1 Chapter Overview

Being able to work conveniently with longer scripts also allows us to solve more complicated problems. But keeping all the ideas in your head for what you want to do on the computer becomes too error-prone as you start dealing with more complicated tasks. The programmer takes on two kinds of work -- the plan for how to solve the problem, and the coding where the plan is communicated to the computer in a programming language. Despite Maple's powerful repertoire of functions (e.g. "solve" or "fit data"), it is often the case that the steps of a plan can require several Maple actions to implement. In those situations it is productive to outline a solution before writing the details of the code. The outline (as well explanations of any tricky details of the code) can be included in the code edit region by including program comments.

The ability to include comments in coding regions also can be used by the programmer as they figure out how to create the code. All the details of the plan need not be made in one fell swoop. A programmer can begin by writing a code outline in comment form. Since the Maple language processor does not perform any actions when it executes a comment, the code edit region will both be helpful to the programmer, and also not contain anything that would cause error messages from Maple.

Once the programmer is satisfied that the outline is satisfactory, the finer details can be filled in much as an outline for a report or paper can be used. When the details are fine enough, the programmer can write the short bits of code that correspond to that detail.

As with all multi-part projects, it is best to develop code incrementally, adding a line or two of code at a time, running the entire region and checking that it works as far as it goes, and then proceeding onto another few lines. The alternative to incremental development -- typing in the entire solution and then trying to fix its problems -- is much more time consuming because there are too many places to look for problems. Incremental development works faster because there are only slight changes from the previously tested version of the code, hence only a few places that could be causing problems if there are any.

11.2 The "outline approach" for developing a solution in a code region.

Many students learn to write a report through the "outline method". See for example, http://owl.english.purdue.edu/owl/resource/544/02/. The outline method suggests that the way to proceed goes roughly like this:

1. After considering what you would like to say, create the major sections for your content, and put them into an order that makes sense.

2. Refine each section, by adding subsections that group the major points of each section. Put the subsections in an order that will make sense to the reader.

3. Continue in this way to create subsubsections, etc. until the level of detail is small enough that you can see an easy writing task for each segment in your outline.

The final version of the report will keep the content organized in this way. This kind of refinement can makes it easier to keep things well-organized and coherent at each level of detail. It also helps avoid getting lost in the fine details.

Code development using this style has the initial outline written as program comments. Only when the subpoints or subsubpoints can be expressed straightforwardly as something the programmer finds easy to code (or one line of code) does the writing turn from writing comments into writing of code.

This makes commenting something you do even before you are completely certain as to what you will do. It's a lot easier to change comments than it is to redo the details of code that has already been written. Having the plan description included into the code region means that you do not have to jump between pieces of paper and the Maple worksheet to check whether the plan is being followed by the code.

Table 11.1: A problem to solve

| Table 11.1: A problem to solve | (From Sullivan, Pre-calculus, p. 342) |
A fast-food restaurant wants a special container to hold coffee. The restaurant wishes the container to quickly cool the coffee from $200^\circ\text{F}$ to $130^\circ\text{F}$ as quickly as possible, and keep the liquid between $110^\circ\text{F}$ and $130^\circ\text{F}$ (optimal drinking temperature) as long as possible. The restaurant has three containers to select from.

(a) The CentiKeeper Company has a container that reduces the temperature of a liquid from $200^\circ\text{F}$ to $100^\circ\text{F}$ in 30 minutes by maintaining a constant temperature of $70^\circ\text{F}$.

(b) The Temp Control Company has a container that reduces the temperature of a liquid from $200^\circ\text{F}$ to $100^\circ\text{F}$ in 25 minutes by maintaining a constant temperature of $60^\circ\text{F}$.

(c) The Hot'n'Cold Company has a container that reduces the temperature of a liquid from 200 to 120 in 20 minutes by maintaining a constant temperature of $65^\circ\text{F}$.

How long does it take each container to lower the coffee temperature from $200^\circ\text{F}$ to $130^\circ\text{F}$?

How long will the coffee temperature remain between $110^\circ\text{F}$ and $130^\circ\text{F}$?

Using the method we suggest, we would first create a code region that has only comments, giving the basic outline of what we would do. In this situation, it would also be good to create a preface describing the problem. In Maple, we could just place the description as ordinary text before the code region with the comments.

We think about what we would do and realize that these questions can be answered by doing the "heating and cooling equation" calculations we've seen previously. Such equations typically require finding a "heating constant" used in an exponential formula, and then solving one or more equations for $t$.

Table 11.2: Initial code outline to solve the problem

<table>
<thead>
<tr>
<th>Initial code outline to solve the problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Description of problem</td>
</tr>
<tr>
<td>(From Sullivan, Pre-calculus, p. 342)</td>
</tr>
</tbody>
</table>

A fast-food restaurant wants a special container to hold coffee. The restaurant wishes the container to quickly cool the coffee from $200^\circ\text{F}$ to $130^\circ\text{F}$ as quickly as possible, and keep the liquid between $110^\circ\text{F}$ and $130^\circ\text{F}$ (optimal drinking temperature) as long as possible. The restaurant has three containers to select from.

(a) The CentiKeeper Company has a container that reduces the temperature of a liquid from $200^\circ\text{F}$ to $100^\circ\text{F}$ in 30 minutes by maintaining a constant temperature of $70^\circ\text{F}$.

(b) The Temp Control Company has a container that reduces the temperature of a liquid from $200^\circ\text{F}$ to $100^\circ\text{F}$ in 25 minutes by maintaining a constant temperature of $60^\circ\text{F}$.

(c) The Hot'n'Cold Company has a container that reduces the temperature of a liquid from 200 to 120 in 20 minutes by maintaining a constant temperature of $65^\circ\text{F}$.

How long does it take each container to lower the coffee temperature from $200^\circ\text{F}$ to $130^\circ\text{F}$?

How long will the coffee temperature remain between $110^\circ\text{F}$ and $130^\circ\text{F}$?

#Define shared parameters for problem
Each of these steps needs to be fleshed out a bit more before we can start writing lines of code. We don't have to flesh out things starting from the beginning, we can add more details to the outline in any appealing order. In this case, we talk about how to answer the questions for CentiKeeper, using the solution process discussed in the outline approach introduction as a model for the computation. We still have only comments, not code. We can use two ##s at the start of the line (which, since it begins with a #, is still regarded as a comment) to indicate that the comment is at a lower level than before.

Table 11.3: Next step of refinement for code outline

<table>
<thead>
<tr>
<th>Next step of refinement for code outline</th>
</tr>
</thead>
<tbody>
<tr>
<td>#Define shared parameters for problem</td>
</tr>
<tr>
<td>#Define parameters for each company</td>
</tr>
<tr>
<td>#Answer the questions for CentiKeeper</td>
</tr>
<tr>
<td>##Find the heating constant k1 for CentiKeeper</td>
</tr>
<tr>
<td>##Substitute the constant into the heating equation.</td>
</tr>
<tr>
<td>##Find the value of t such that the heat equation produces 130 degrees.</td>
</tr>
<tr>
<td>##Find the value of t such that the heat equation produces 110 degrees.</td>
</tr>
<tr>
<td>##Compute the difference, print out the answer.</td>
</tr>
<tr>
<td>#Answer the questions for Temp Control</td>
</tr>
<tr>
<td>#Answer the questions for Hot'n'Cold</td>
</tr>
</tbody>
</table>

The general heating formula we're going to use is \( u(t) = T + (u_0 - T) e^{kt} \). \( u_0 \) is the initial temperature, and \( T \) is "room temperature". \( k \) is the heating constant. We realize that there are other entities too which we give mathematical names to: \( \tau \), a period of time elapsed after the start, and \( N \), the temperature of the liquid in the cup after \( \tau \) minutes has elapsed. We will use different \( u_0 \), \( T \), \( \tau \), \( N \) and \( k \) for each company. We will give different names to the programming variables that we will use for each company, rather than trying to re-use the symbols several times.

We can flesh out the outline further with these decisions.

Table 11.4: Next step of refinement for code outline

<table>
<thead>
<tr>
<th>Next step of refinement for code outline</th>
</tr>
</thead>
<tbody>
<tr>
<td>#Define shared parameters for problem</td>
</tr>
<tr>
<td>##There are no shared parameters</td>
</tr>
</tbody>
</table>
# Define parameters for each company
## u01, T1, k1, tau1, N1 for CentiKeeper
## u02, T2, k2, tau2, N2 for Temp Control
## u03, T3, k3, tau3, N3 for Hot'n'Cold

## Answer the questions for CentiKeeper
## Find the heating constant k1 for CentiKeeper
## Substitute the constant into the heating equation.
## Find the value of t such that the heat equation produces 130 degrees.
## Find the value of t such that the heat equation produces 110 degrees.
## Compute the difference, print out the answer.

## Answer the questions for Temp Control
## Do the same thing as we did for CentiKeeper, only use the variables we have for Temp Control

## Answer the questions for Hot'n'Cold
## Do the same thing as we did for CentiKeeper, only use the variables we have for Hot'n'Cold

We still haven't written any code, but we are about ready to do that. First however, we should learn a few more "tricks" to efficient code development.

### 11.3 A crucial skill in debugging: identify the first line of code where things go wrong

The execution trace of a code region is the output that results from executing each line of a block of code in a region. Full execution traces can be created only after the syntax errors have been removed from the code region so that the region can be executed without any error messages. Removing error messages is only the beginning to the work, however. The code you write may still have mistakes in it -- the instructions you give may make grammatical sense, but be the wrong actions to solve the problem. A mistake in code is often called a bug, and removing mistakes from code is called debugging it. Programmers spend a lot of time debugging, usually because getting something new to work is difficult. Any ideas, approaches, or strategies for making debugging less difficult are usually welcomed by programmers.

#### Using an execution code trace to find a bug in a short code block

```plaintext
# Define parameters for each company
## u01, T1, k1, tau1, N1 for CentiKeeper

u01 := 200;
T1 := 70;
N1 := 100;
tau1 := 30;

200
T1 = 70
100
30
```

(11.1)
After creating the code region and entering the lines, we get the execution trace  --  the lines in blue  --  by clicking on the code edit region and typing either control-e (command-e for Macintosh), or entering right-click->Execute Code. Each line in blue corresponds to result of execution of a line of code.

A major difference between executing a script interactively in a document, and in a code edit region, is that execution trace for interactive execution occurs intermingled with the lines of code, while with code regions the execution trace occurs after the listing of all the lines of the block.

Programmers debugging code inspect execution traces to find mistakes in their programming. They try to find the first line of code where things go wrong. While the cause of that "buggy result" may lie in something that happened even earlier, it is certain that the first program bug did not happen later. Knowing the first buggy line of the execution trace typically reduces the number of lines of code that must be inspected for the mistake, since the error must have occurred in the code at or before the line that generated the "bad output".

In the execution trace above, we see that the first line of output is consistent with the "u01 := 200;" line of code -- the value of the assigned variable is printed out. We can see immediately that the T1=70 line looks different than the others, which were just numbers from the assignments. Looking at the code, we see that there is a difference between the second line of code (excluding comments) and the others -- there is an "=" rather than a ":=". We have used the execution trace to discover a mistake we have made in a line of code -- that line should really be T1 := 70.

### 11.4 Using print and printf to include more information in the execution trace

Sometimes the execution trace does not provide enough information to discover a bug. For that reason, programmers sometimes rely on placing print or printf statements in code regions just before steps that they want to look at. Usually the statements that do the printing output a message that includes not just values of variables, but what variables they are, and other information that makes it easier to identify which line of code out of many in the code block is doing the printing.

Continuing the example above, we enter more lines of code and begin to execute what we have. We ran into problems where some of the variables got mangled from bugs in previous executions of the code region. So we place a restart at the beginning to make sure that we unassign all variables before we begin execution again, leaving only the assignments made this time around.

We put in a printf statement to make it clear what we're solving for, before we solve it. We do this because we know (from previous efforts working with this kind of problem) that solving this "heat equation" can be tricky.

Table 11.5: Execution trace augmented by a printf

```plaintext
#Define parameters for each company
##u01, T1, k1, tau1, N1 for CentiKeeper
restart;  #Unassign variables from previous executions
u01 := 200;
T1 := 70;
N1 := 100;
tau1 := 30;

#Answer the questions for CentiKeeper
##Find the heating constant k1 for CentiKeeper

###Describe the heat equation
heat1 := T1+(u01-T1)*exp(k1*t);
ht1 := eval(heat1,t=tau1);
printf("Solving for %a",ht1);
k1soln := solve(ht1,k1);
printf("Solving %a for heat constant: k1 is %f", ht1, k1soln);
```
Solving for \(70 + 130 \cdot e^{30 \cdot k1}\)

\[
\frac{1}{30} \ln \left( \frac{7}{13} \right) + \frac{1}{30} \cdot 1\pi
\]

Solving \(70 + 130 \cdot e^{30 \cdot k1}\) for heat constant: \(k1\) is

Error, (in fprintf) number expected for floating point format

We see a lot of gibberish after awhile, but our attention turns to the "Solving for \(70 + 130 \cdot e^{30 \cdot k1}\)" and the next line of the execution trace, which we infer from its placement is the result assigned to \(k1\) soln. It appears to be an imaginary number (because it has an "I" in it), which is definitely not the right kind of value for a cooling constant. Looking again at the printf output, we see that what we're solving for is wrong -- according to the math we should be solving the equation \(100 = 70 + 130 \cdot e^{30 \cdot k1}\). The Error, message is triggered by the result of solve. fprintf can't handle expressions with I in them which is what the buggy computation is producing. We use these insights (which come only after we understand what we would be doing by hand with this calculation), to rework the code region so that it sets up an equation that uses the "N1" information.

### Table 11.6: Execution trace from a repaired code region

<table>
<thead>
<tr>
<th>Execution trace from a repaired code region</th>
</tr>
</thead>
<tbody>
<tr>
<td>#Define parameters for each company</td>
</tr>
<tr>
<td>##u01, T1, k1, tau1, N1 for CentiKeeper</td>
</tr>
<tr>
<td>restart; #Unassign variables from previous executions</td>
</tr>
<tr>
<td>u01 := 200;</td>
</tr>
<tr>
<td>T1 := 70;</td>
</tr>
<tr>
<td>N1 := 100;</td>
</tr>
<tr>
<td>tau1 := 30;</td>
</tr>
<tr>
<td>##u02, T2, k2, tau2, N2 for</td>
</tr>
<tr>
<td>#Answer the questions for CentiKeeper</td>
</tr>
<tr>
<td>##Find the heating constant k1 for CentiKeeper</td>
</tr>
<tr>
<td>heat1 := N1 - T1+(u01-T1)<em>exp(k1</em>t);</td>
</tr>
<tr>
<td>h1 := eval(heat1, t=tau1);</td>
</tr>
<tr>
<td>printf(&quot;Solving for %a., h1);</td>
</tr>
<tr>
<td>k1soln := solve(h1, k1);</td>
</tr>
<tr>
<td>printf(&quot;Solving %a for heat constant: k1 is %f., h1, k1soln);</td>
</tr>
</tbody>
</table>
Solving for $100 = 70 + 130e^{30k1}$.

This presents information in the trace that convinces us that the right thing is going on.

## 11.5 Incremental coding and testing

The old adage "don't bite off more than you can chew" also applies to coding. Suppose you enter thirty lines of code and then execute them. If you find a mistake at the end, there are thirty lines that could be the source of the problem. Having to look through thirty of anything (the execution trace, or the code) can be time consuming and complicated to keep track of. You may have noticed that we have developed code in the past few sections only a few lines at a time -- entering them, viewing the execution trace, finding things to fix, and proceeding. It would be nice if we could just get everything written correctly the first time, but it is more realistic to assume that you won't. There will be typos and misconceptions to fix all along the way.

Proceeding incrementally makes it likely that it's always the last lines you entered where the problem is most likely to lie. Combining this with information from `print` statements to augment the execution trace can be a very efficient way of developing code because the execution trace information is presented intelligibly, and its typically only the last few lines that you need to scrutinize carefully.

Finally, you can use the "comment trick" to handle things incrementally even if you've already written more than two lines of code. You can type in all the lines of code, but add a `#` in front of all the lines except the first one. If you then execute the code region, you will see the results from only the first line of code because all the others will be treated as comments and not executed. Once you have that line working well, then you can remove the `#` from the second line, and re-execute the region. Because two lines are now not comments, you will see the execution trace from two lines of code. You can continue "uncommenting" lines in this fashion to achieve incremental development.

The comment trick can be used to on `printf` statements that you have been using for troubleshooting but are not completely sure that you no longer need. By putting a `#` in front of a print statement, you can turn off its effects without erasing it completely. If you decide that you need to see it again, you can remove the `#` and re-activate it. This can save a lot of typing.

Having proceeded to solve one version of the problem partially, let's see if we can create the whole ball of wax for CentiKeeper. We enter one new line at a time, execute the region, and see what needs fixing. After about five minutes, we have all the lines working, and have printed the desired answer at the end. Note that we have used an advanced feature of so that it prints only two digits after the decimal point. We also rely on the fact that even if the value being given to a `%f` format code isn't a floating point number, the result is automatically approximated so that it can appear in floating point format. You can read about these feature in the on-line help for `printf`.

At this point, we'll clone the process for the other two situations through copying, pasting, and editing to use different variable names.
Table 11.7: Execution trace augmented by a *printf*

<table>
<thead>
<tr>
<th>Execution trace augmented by a <em>printf</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>#Define parameters for each company</td>
</tr>
<tr>
<td>#u01, T1, k1, tau1, N1 for CentiKeeper</td>
</tr>
<tr>
<td>restart; #Unassign variables from previous executions</td>
</tr>
<tr>
<td>u01 := 200;</td>
</tr>
<tr>
<td>T1 := 70;</td>
</tr>
<tr>
<td>N1 := 100;</td>
</tr>
<tr>
<td>tau1 := 30;</td>
</tr>
</tbody>
</table>

#Answer the questions for CentiKeeper

##Find the heating constant k1 for CentiKeeper

###Describe the heat equation

heat1 := T1+(u01-T1)*exp(k1*t);
ht1 := N1 =eval(heat1,t=tau1);
printf("Solving for %a.",ht1);
k1soln := solve(ht1,k1);
printf("Solving %a for heat constant: k1 is %f.", ht1, k1soln);

##Substitute the constant into the heating expression.
eqn1 := eval(heat1, k1=k1soln);
##Find the value of t such that the heat equation
##produces 130 degrees.
CentiKeeper130 := solve(eqn1=130,t);

##Find the value of t such that the heat equation
##produces 110 degrees.
CentiKeeper110 := solve(eqn1=110,t);

##Compute the difference, print out the answer.
printf("The CentiKeeper cup keeps coffee warm for %5.2f minutes.",
    CentiKeeper110-CentiKeeper130);
Solving for $100 = 70 + 130 \cdot e^{30k_1}$.

$$100 = 70 + 130 e^{30k_1}$$

We solve for $k_1$:

$$\frac{1}{30} \ln \left( \frac{3}{13} \right)$$

Solving $100 = 70 + 130 \cdot e^{30k_1}$ for heat constant: $k_1$ is $-0.048878$.

$$70 + 130 e^{\frac{1}{30} \ln \left( \frac{3}{13} \right)}$$

$$\frac{30 \ln \left( \frac{6}{13} \right)}{\ln \left( \frac{3}{13} \right)}$$

$$\frac{30 \ln \left( \frac{4}{13} \right)}{\ln \left( \frac{3}{13} \right)}$$

The CentiKeeper cup keeps coffee warm for 8.30 minutes.

### 11.6 Using interactive development first for tricky code

We've already mentioned an idea to speed up development of code that you're not completely sure about -- copy examples from documentation and get them to work in a worksheet, then modify the working example to do what you really would like to do. When developing a sizable script in a code region, there will still be spots where you won't know what to enter immediately. When you encounter such a situation, open a fresh Maple document (i.e. by doing File->New->Document Mode) and use it as an experimental sandbox working interactively to get those lines of code to work. Then copy the working code into the code region when it is "fully baked" or nearly so. If you have developed the code interactively using 2D math notation, you may need to convert it into ordinary textual format by applying Format->Convert To->1D Math Input.

### Converting 2D expressions into 1D expressions

We enter an expression in the usual interactive two dimensional format. It's probably easier to enter the expression this way and see that it agrees with the mathematical conception.

We copy the expression and paste it into a separate place on the worksheet. We then select the expression and do Format->Convert To->1D Math Input. The result in red replaces the copy of the expression. We can then copy this into a code region.
We create a function definition involving this expression by entering the function name, the assignment operation :=, and the beginning of the function definition. Then we copy and paste in the 1D expression that we've manufactured.

\[
x_{\text{posWind}} := (t) \rightarrow x_{0}-m v_{0x} \left( -1 + \exp \left( -\frac{b t}{m} \right) \right)/b;
\]

(11.6)

### 11.7 Preventing catastrophic loss of worksheet information

The advice is: Save the current state of your work after making significant changes to the content of your worksheet.

When you start working for longer periods of time with computers, the unexpected may become more likely. Several of these unexpected scenarios could cause your work to be seriously disrupted.

1. There could be a power failure, or a system crash or freeze.

Saving your document before embarking on a test run will at least allow you to recover what you had before the crash. Crashes or freezes are not totally unheard in highly interactive programs with complicated interfaces, such as web browsers, Java-based applets, or systems such as Maple (whose interface is Java based). While you can try to recover from what Maple saved automatically while you were working (see the discussion in this section (page 20)), saving a file yourself gives you explicit control over what you will recover after the crash.

2. Your most recent changes to your work are a big mistake. Suddenly you wish you could revert back to the way things were an hour ago, or yesterday.

Unless you are working in an environment that automatically preserves successive versions of your work, you will have to keep copies the old fashioned way -- by saving them under different names. One simple way is to append a version number, e.g. MyLab1-1.mw, MyLab1-2.mw, etc. If you aren't up to keeping track of the latest version number yourself, you could also use a time index label, e.g. MyLab1-103109-1014.mw.

3. Your computer could break down, making the files on it unavailable until the computer repaired, or even permanently unreadable should repairs not be possible. Computer theft is another way of losing the files on that computer permanently.

This doesn't happen all that often, but probably happens to everyone who uses a computer for a few years. Hopefully you have saved copies of your valuable files elsewhere. This process is called creating a backup copy of your files, or just backing up. In this era of "cloud computing" and inexpensive external storage (writable DVDs, external flash or hard drives), this is fairly easy to do for those who have the discipline to back up regularly. There are even some commercial programs (e.g. Mozy https://mozy.com/registration, SpiderOak https://spideroak.com/, or Macintosh Time Machine http://www.apple.com/macosx/what-is-macosx/time-machine.html, for example) that create "back up copies" automatically at periodic intervals without any effort on your part other than to make sure that the computer is powered on when the back ups should happen. The main point is to have the discipline to create backups before you end up in a situation where you have to work many weeks to recreate lost work.

Creating and having convenient access to back ups is expected for professionals who use computers. Losing the most recent version of a file may happen to everyone occasionally; losing a valuable file completely is as lame an excuse for a professional to use as "I forgot to do my homework" is for a student barring true calamities such as a natural disaster.
### 11.8 Summary of Chapter 11

<table>
<thead>
<tr>
<th>Code Development Strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>Create a code outline as a series of comments in a code region before you start coding. Refine the outline by adding subpoints until you have supplied enough details to write code.</td>
</tr>
<tr>
<td>Add <code>print</code> or <code>printf</code> statements to code in order to make the execution trace more intelligible and to make key results easier to find within it.</td>
</tr>
<tr>
<td>The goal in debugging is typically to inspect the execution trace and find the first place where things go wrong. Use this information to determine the line(s) of code you should look at to find and fix the causes of problems. In order to do this you need to know what is right, and you should create situations where you don't have to look in too many places.</td>
</tr>
<tr>
<td>Develop incrementally -- enter only a few lines of code at a time, and get them to work before adding more. Since comments always work, you can enter entire outlines without having to be particularly incremental.</td>
</tr>
<tr>
<td>Use an interactive worksheet to test out small segments of code before you enter them into a worksheet. This can be an easy way of entering a mathematical expression, for example.</td>
</tr>
<tr>
<td>Save your work to avoid losing it. Save your work using versioned names in order to avoid having new work mess up clean copies of things that used to work. Save your work on other computers to avoid losing it completely should your computer break.</td>
</tr>
</tbody>
</table>
12 Tables and other data structures

12.1 Chapter Overview

This brief chapter introduces Maple tables. Tables are another way in Maple to store multiple values in a single container. Like most other structures in Maple, they can hold any number of items.

12.2 A review of Maple data structures and properties

The value of computers in technical work comes not only from doing long calculations conveniently, but being able to do thousands or millions of shorter computations quickly. In order to harness this power without the programming becoming very tedious, we need to store inputs and outputs in data containers or data structures. We've already seen lists and sets as containers. Maple also has specialized containers used for particular purposes: ranges, equations, and expressions.

Let's review the data containers we've seen and used in Maple so far. Each kind of container has a different syntax to create it, and to access its parts. Variables can be assigned as their value their whole container. Some Maple functions use the whole container, while in other cases you use "parts access" to use particular items within the container. One common aspect of these data containers is that they are immutable: once you've created them, you can't change them, you can only access (find out the value of) their value. In order to "change" a list or expression, for example, what you have to do is to create a new list/expression with the changes in it, and assign it to the same variable as had the old list.

<table>
<thead>
<tr>
<th>Type of container</th>
<th>Examples</th>
<th>How to create it</th>
<th>How to access a part of it</th>
<th>Operations that can be performed with container</th>
<th>Ways of changing data container after it's been created</th>
</tr>
</thead>
<tbody>
<tr>
<td>Math expression</td>
<td>a+b+c</td>
<td>Type it in using a arithmetic operation or mathematical function</td>
<td>op(i, expr) gets the i-th piece of the expression</td>
<td>Arithmetic and math functions: +, -, *, ^, sqrt, !, etc., nops(expr) tells you the number of terms in the expression</td>
<td>Can't do it. Can only create a new container with edited content.</td>
</tr>
<tr>
<td>Range</td>
<td>1..10</td>
<td>Type it in using a &quot;..&quot;</td>
<td>op(1, r) gets the low end of the range. op(2, r) gets the high end</td>
<td>None other than access functions, but many functions use ranges as one of their parameters.</td>
<td>Can't do it.</td>
</tr>
<tr>
<td>Equation</td>
<td>2<em>x-3=5 as in solve(2</em>x-3=5, x)</td>
<td>Type it in using a &quot;=&quot;, &quot;=&quot;</td>
<td>rhs(eqn), lhs(eqn) gets the right and left hand sides of the equation</td>
<td>solve, fsolve, plot, etc. although usually other arguments are necessary</td>
<td>Can't do it.</td>
</tr>
<tr>
<td>List</td>
<td>[1,1,2,3,5.5]</td>
<td>Type in items separated by commas, enclosed by square brackets.</td>
<td>L[i] gets the i-th item from the list. If i is negative, counting starts from the end. L[i..j] gets a list of the ith through jth items.</td>
<td>nops tells you the number of items in the list</td>
<td>Can't do it.</td>
</tr>
<tr>
<td>Sequences</td>
<td>1,2,3,5,5 NULL #empty sequence</td>
<td>Type in items, separated by commas.</td>
<td>L[i] gets the i-th item from the list. If i is negative, counting starts from the end. L[i..j] gets a list of the ith through jth items.</td>
<td>All Maple functions accept sequences, although not all of them work with sequences that are the wrong length or provide the wrong kinds of values.</td>
<td>Can't do it.</td>
</tr>
<tr>
<td>Set</td>
<td>{1,2,3,5.5} {} #empty set</td>
<td>Type it in: items separated by commas, enclosed by curly brackets.</td>
<td>S[i] gets the i-th item from the set.</td>
<td>nops tells you the number of items in the list union, intersect, minus, perform set operations</td>
<td>Can't do it.</td>
</tr>
</tbody>
</table>
12.3 Tables

In this section, we introduce another kind of data structure, tables. A table is a structure where each element is accessed using a value called a key or index. Individual elements of the table can be assigned, in any order. Individual elements can be accessed using the index notation, e.g. $t[2]$ to get the element of the table $t$ whose access key is 2. This is similar to how lists worked, where $L[2]$ for a list $L$ was an expression whose value was the 2nd element of the list $L$.

The key can be any valid Maple value, not necessarily an integer (although they often are). Tables can add elements over time with additional assignments.

Tables are useful in situations where we will be adding items to the data structure frequently, but we don't know in advance how many entries we will ultimately have. They will be more efficient to use than lists or sets in that kind of situation. In future chapters we will also see ways to exploit the other useful property of tables, where it is more convenient to have non-integer values for access keys.

---

**Example Demonstration of tables**

```maple
restart
t := table( )
t := table( ) # (12.1)
T[1] := 3.5
T[1] := 3.5 # (12.2)
T[2] := 4.6
T[2] := 4.6 # (12.3)
T[4] := 17.3 # (12.4)
T[2] := 4.7 # (12.5)
T[-1]
T[-1] # (12.6)
indices(T)
indices(T) # (12.7)
```

This creates a table named $T$. Initially there are no items in the table. Whatever previous value $T$ may have had is wiped out.

Through assignment we create a table entry for $T[1]$.

Through assignment we create a table entry for $T[2]$.


We reassign the value of $T[2]$, overwriting the previous value.

Expressions mentioning assigned elements of the table will use the assigned value. If that element of the table does not have an assigned value, it remains just as a symbol.

The `indices` function will return a sequence of the access keys for the table. Each access key is enclosed in brackets so that you can tell where one key stops and the next begins.
There are three major differences between tables in Maple and the data containers we've seen previously: a) there is a separate creation step (creating a table), b) you can change the contents of the table be assigning, c) items within a table are stored under a key or index value. The latter works similar to the indexing used for lists, but the index does not have to be an integer. A table can store any kind of item under any kind of key. For the time being, we will stick with integer keys, but eventually we will take greater advantage of this "key" feature.
12.4 Troubleshooting with tables

1. Table creation $tbl := \text{table}(\text{);}$ should always appear at the beginning of a script, which assigns $tbl$ a fresh, empty table. If you don't do this, you may see items left over in $tbl$ from the last time you executed a script in the same worksheet with $tbl$ in it. Alternatively, you can do a \textit{restart}; before running a script that uses a table.

2. Use entries and indices to see what's in your table. Check the results to make sure that there are no unexpected indices and that the values are computed correctly.

3. Know when to use square braces and when to use parentheses in working with tables.

**Troubleshooting with tables: mistaking $[ ]$ and $()$**

<table>
<thead>
<tr>
<th>$T := \text{table}(\text{);}$</th>
<th>\textit{table$([\text{ ]})$}</th>
<th>(12.20)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T(3) := 4.7$</td>
<td>$4.7$</td>
<td>(12.21)</td>
</tr>
<tr>
<td>Error, invalid left hand side in assignment</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Accessing a table value requires the use of square brackets $[ ]$ to enclose the index. Use of parentheses makes Maple think you are talking about functions rather than tables.

| $T[3] := 4.7$               | $4.7$                       | (12.22) |
| $T(3)$                      | $\text{table([3 = 4.7])(3)}$ | (12.23) |
| $T[3]$                      | $4.7$                       | (12.24) |

We place an entry into the table -- the value 4.7 accessed by the key 3.

This does not access values in the table. It uses $(\text{ )}$s rather than the necessary $[\text{ ]}$s.

**Troubleshooting tables: checking for extra keys**

<table>
<thead>
<tr>
<th>$T := \text{table}(\text{);}$</th>
<th>\textit{table$([\text{ ]})$}</th>
<th>(12.24)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T[3] := 4.9$</td>
<td>$4.9$</td>
<td>(12.27)</td>
</tr>
<tr>
<td>$T[3]$</td>
<td>$4.9$</td>
<td>(12.28)</td>
</tr>
</tbody>
</table>

This a demonstration of how the fact that "a table key can be any value" can create confusion. We create a table and place three entries into it.
### Troubleshooting tables: checking for extra keys

We assign \( i, j, \) and \( k \) the values of 2, 3.0 and 4.

We expect that this will result in creating or changing certain entries of the table.

\[
i := 2
\]

\[
k := i + 1.0
\]

\[
j := 6 - i
\]

\[
T[k] := 5.0
\]

\[
T[j] := 5.1
\]

\[
T[1] := 4.75
\]

\[
T[1], T[2], T[3], T[4]
\]

Where did the 5.0 go? We thought we did \( T[3.0] := 5.0 \) with the use of \( k \).

We use the \textit{indices} and \textit{entries} functions to tell us what is going on in \( T \). Evidently the keys used for \( T \) include both 3 and 3.0. Dimly, light comes on where we recall that 3 and 3.0 are regarded as different in Maple. "3" is an exact integer and "3.0" is a floating point number.

We see that there are two distinct entries in the table. Reassigning one of them does not erase the other entry, although the value of the first one does change.

\[
\text{indices}(T)
\]

\[
[1], [2], [3], [4], [3.0]
\]

\[
\text{entries}(T)
\]

\[
[4.75], [4.8], [4.9], [5.1], [5.0]
\]
Troubleshooting tables: checking for extra keys

\[ T[3.0] := 'T[3.0]' \]

As indicated in the demonstration of tables (page 185), unassigning a table entry will erase it from the table.

\[ T_{3,0} \]

\[ \text{indices}(T) \]

\[ [1], [2], [3], [4] \]

\[ \text{entries}(T) \]

\[ [4.75], [4.8], [5.0], [5.1] \]

\[ T[3.0] \]

\[ T_{3,0} \]

(12.43)

(12.44)

(12.45)

(12.46)

12.5 Summary of Chapter 12

<table>
<thead>
<tr>
<th>Type of container</th>
<th>Examples</th>
<th>How to create it</th>
<th>How to access a part of it</th>
<th>Operations that can be performed with container</th>
<th>Ways of changing data container after it's been created</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table</td>
<td>T := table(); tblName := table(); T[1] := 32.5; T[2] := 32.6;</td>
<td>This creates an empty table and assigns it to the variable tblName replacing whatever previous value it had.</td>
<td>T[key] retrieves the value stored under the key from within the table. Thus if you have done T[1] := 32.5; then subsequently doing T[1] will retrieve the 32.5. The key can be any value Maple expression. Thus x, or ( t=3..5 ) can be keys for a Table</td>
<td>entries(T) provides a sequence of the values stored in the table (including duplicate values) indices(T) provides a sequence of the keys used by the table.</td>
<td>Key-value assignment: T[1] := 32.7</td>
</tr>
</tbody>
</table>
13 For Looping: repetitive execution

13.1 Chapter Overview

The computer can carry out thousands or millions of instructions in Maple per second. While a "powerful" instruction such as solve or plot to Maple is a way of having a few lines of code tap into this power, we need a way of specifying calculations that perform a lot of work without writing a commensurate amount of code. Maple's for...do...end do and while...do...end do instructions are a way of specifying that lines of code be repeated a definite number of times, or indefinitely until some condition is no longer true. This chapter describes the syntax -- how to say it -- and some example calculations that use this capability.

Such programming is called looping because the computer is told to loop back and repeat some instructions. Loop programming is a basic skill all programmers need to know, since most programming languages have similar ways of doing repetition.

13.2 Repetition with to ... do .... end do

In programming it's common to want to repeat a segment of a code region a few times. We could do that by copying and pasting the code that many times, but this gets tedious. We might also want to place the repetition factor under control of a script parameter, so that it would be easy to change the number of times things could be repeated.

Maple provides a convenient way to control the repetition of a code segment through something called a loop. The segment of code that is repeated is called the loop body. Its general form looks like this:

to n do
  segment of code
end do;

where \( n \) is an expression describing how many times the segment should be repeated.

**Table 13.1: Examples of repetitive looping with to...do...end do**

<table>
<thead>
<tr>
<th>Examples of repetitive looping with to...do...end do</th>
<th>The 'n at the end of the printf is necessary or else all the output will appear on the same line.</th>
</tr>
</thead>
</table>
| to 5 do
  printf("I will not burp in class.\n");
  end do;
  printf("The repetition is over.\n"): | Recall that the execution trace contains both the results of each line execution, and the output of any print or printf statements that are being executed. Since the ??? which displays as "nothing", this execution trace contains just the side effect of displaying the various messages. |
| I will not not burp in class. | (For these and other repetitive sayings by Bart Simpson, see http://www.snpp.com/guides/chalkboard.openings.html.) |
| I will not not burp in class. | This script makes it easy to change the number of times the loop repeats itself, through the standard |
| I will not not burp in class. | |
| I will not not burp in class. | |
| The repetition is over. | |

\[ n := 7; \to n \do \newline \text{printf("I will write \$d times, "Underwear should be worn on the inside.\"\n", n);} \newline \text{end do;} \newline \text{printf("The repetition is over.\n")}; \]
7

I will write 7 times, "Underwear should be worn on the inside."
I will write 7 times, "Underwear should be worn on the inside."
I will write 7 times, "Underwear should be worn on the inside."
I will write 7 times, "Underwear should be worn on the inside."
I will write 7 times, "Underwear should be worn on the inside."
I will write 7 times, "Underwear should be worn on the inside."
I will write 7 times, "Underwear should be worn on the inside."
The repetition is over.

Troubleshooting loops: if the "end do" is left out - - you may see a warning and more importantly, none of the expected repetitions. Furthermore, none of the post-repetition actions seem to have been performed.

n := 7;
to n do
  printf("I will write %d times, \"Underwear should be worn on the inside.\"\n", n);
  printf("The repetition is over.\n");
end do;
Warning, premature end of input, use <Shift> + <Enter> to avoid this message.

13.3 Repetition with for ... to ... do ... end do and for ... to ... by .... do ... end do

We can introduce a "stepper variable" into the repetition through the use

for stepperVariable from initial value to final value do .... end do;

Sometimes the stepper variable is referred to as an index variable.

The first time the code segment is executed, the value of the stepper variable is initial value.
The second time, it is set to one more, initial value + 1. etc. The last repetition is typically final value.

Typically, initialValue and finalValue are expressions that evaluate to integers (whole numbers). If initialValue is omitted, then by default it is 1.

Table 13.2: Example of repetition with a stepper/index variable: for ... from .. to ... do ... end do

Example of repetition with a stepper/index variable: for ... from .. to ... do ... end do

Since the initial value is omitted, the default value of 1 for initialValue is used here,

Maple, having Algol 68 in its ancestry, allows loop bodies to end with "od" (do spelled backwards) as well as the more English like "end do". Either one is equally acceptable.
We can make the stepperVariable take on different values between initialValue and finalValue through the use of a stepValue. for stepperVariable from initialValue to finalValue by stepValue do ... end do will repeat the loop.

Table 13.3: Example of repetition with a stepper/index variable: for ... from .. to .. by ... do ... end do

<table>
<thead>
<tr>
<th>Example of repetition with a stepper/index variable: for ... from .. to .. by ... do ... end do</th>
<th>This kind of loop can be used to print out tables of values, where each row depends on the value of the stepper variable. What would we have to change in order to print out values of sine for 0.0, 0.05, 0.10, 0.15?</th>
</tr>
</thead>
<tbody>
<tr>
<td>#Final value parameter</td>
<td></td>
</tr>
<tr>
<td>fv := 1.2;</td>
<td></td>
</tr>
<tr>
<td>#Repeat an action with i stepping through the values</td>
<td></td>
</tr>
<tr>
<td>#0, .1, .2, .3, ....</td>
<td></td>
</tr>
<tr>
<td>for x from 0 by .1 to fv do</td>
<td></td>
</tr>
<tr>
<td>printf(&quot;sin(%f) = %f.\n&quot;, x, evalf(sin(x))) end do;</td>
<td></td>
</tr>
<tr>
<td>printf(&quot;The repetition is over.\n&quot;);</td>
<td></td>
</tr>
<tr>
<td>sin(0.000000) = 0.000000.</td>
<td></td>
</tr>
<tr>
<td>sin(0.100000) = 0.099833.</td>
<td></td>
</tr>
<tr>
<td>sin(0.200000) = 0.198669.</td>
<td></td>
</tr>
<tr>
<td>sin(0.300000) = 0.295520.</td>
<td></td>
</tr>
<tr>
<td>sin(0.400000) = 0.389418.</td>
<td></td>
</tr>
<tr>
<td>sin(0.500000) = 0.479426.</td>
<td></td>
</tr>
<tr>
<td>sin(0.600000) = 0.564642.</td>
<td></td>
</tr>
<tr>
<td>sin(0.700000) = 0.644218.</td>
<td></td>
</tr>
<tr>
<td>sin(0.800000) = 0.717356.</td>
<td></td>
</tr>
<tr>
<td>sin(0.900000) = 0.783327.</td>
<td></td>
</tr>
<tr>
<td>sin(1.000000) = 0.841471.</td>
<td></td>
</tr>
<tr>
<td>sin(1.100000) = 0.891207.</td>
<td></td>
</tr>
<tr>
<td>sin(1.200000) = 0.932039.</td>
<td></td>
</tr>
<tr>
<td>The repetition is over.</td>
<td></td>
</tr>
</tbody>
</table>

#Final value parameter

fv := 1.2;

#Repeat an action with i stepping through the values

#0, .1, .2, .3, ....

for x from fv to 0 by -.1 do

printf("exp(%f) = %f.\n", x, evalf(exp(x))) end do;

printf("The repetition is over.\n");

We can make the stepper value decrease in values with each repetition by having the starting value (in this case, 1.2) larger than the final value (in this case, 0), and making the step value negative (-.1).
\[ \exp(1.200000) = 3.320117. \]
\[ \exp(1.100000) = 3.004166. \]
\[ \exp(1.000000) = 2.718282. \]
\[ \exp(0.900000) = 2.459603. \]
\[ \exp(0.800000) = 2.225541. \]
\[ \exp(0.700000) = 2.013753. \]
\[ \exp(0.600000) = 1.822119. \]
\[ \exp(0.500000) = 1.648721. \]
\[ \exp(0.400000) = 1.491825. \]
\[ \exp(0.300000) = 1.349859. \]
\[ \exp(0.200000) = 1.221403. \]
\[ \exp(0.100000) = 1.105171. \]
\[ \exp(0.000000) = 1.000000. \]

### 13.4 Repetition with for... from ... to ... by .... do ... end do

General form:

\[
\text{for } \text{stepperVariable} \text{ from } \text{initialValue} \text{ to } \text{finalValue} \text{ by } \text{stepValue} \text{ do } \\
\text{loop body;} \\
\text{end do;}
\]

We can make the stepperVariable take on different values between initialValue and finalValue through the use of a stepValue. We can even make the stepperValue decrease rather than increase by giving a negative value for the stepValue.

#### For loop with a different step size

```
for x from 0 by .1 to 1 do
    printf("\sin(\%f) = \%f.\n", x, evalf(sin(x)));
end do;
```

<table>
<thead>
<tr>
<th>Value</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>sin(0.000000)</td>
<td>0.000000</td>
</tr>
<tr>
<td>sin(0.100000)</td>
<td>0.099833</td>
</tr>
<tr>
<td>sin(0.200000)</td>
<td>0.198669</td>
</tr>
<tr>
<td>sin(0.300000)</td>
<td>0.295520</td>
</tr>
<tr>
<td>sin(0.400000)</td>
<td>0.389418</td>
</tr>
<tr>
<td>sin(0.500000)</td>
<td>0.479426</td>
</tr>
<tr>
<td>sin(0.600000)</td>
<td>0.564642</td>
</tr>
<tr>
<td>sin(0.700000)</td>
<td>0.644218</td>
</tr>
<tr>
<td>sin(0.800000)</td>
<td>0.717356</td>
</tr>
<tr>
<td>sin(0.900000)</td>
<td>0.783327</td>
</tr>
<tr>
<td>sin(1.000000)</td>
<td>0.841471</td>
</tr>
</tbody>
</table>
```

The from, to, and by portions of a for are all optional. If that portion is skipped, then 1 is used for the initialValue, finalValue, or stepValue. We can give the information in any order and it will mean the same thing. Thus "for i from 1 to 10 by 3..." is as acceptable as "for i to 10 by 3 from 1 do".

#### For loop with a decrementing rather than incrementing steps

```
for x by -.3 to -1 do
    printf("\sqrt(\%f) = %.a\n", x, evalf(sqrt(x)));
end do;
```
We are using %a format rather than %e or %f format because some of the answers are imaginary numbers, which are not floating point so don't work with the other formats.

Since there is no "from" in the for statement, the initial value of x is 1. Since the step size is -.3, then the finalValue should be less than initialValue of 1.

### 13.5 A pattern of looping: accumulation of results

So far we have seen examples of loops where what happens inside the loop is based on the current value of the index. However, by doing a little extra work before and during the loop, we can calculate accumulative results.

For example, suppose we want to add together the first 100 numbers.

We could start typing in the single expression:

```plaintext
result := 1 + 2 + 3 + 4 + 5 +
```

but we can see that this is going to be a lot of work. We can do this with a loop by using the ability of variables to add onto their present value. What is required to initialize the variable, and then repetitively accumulate.

Table 13.4: An example of an accumulation loop

```plaintext
#Set parameters of script
n := 10;

#initialize accumulation variable
result := 0;

#repeat: add i to the accumulated sum
for i to n do
    result := result + i;
end do;

printf("The sum of the numbers from 1 to %d is: %d", n, result);
```
The sum of the numbers from 1 to 10 is: 55

We see that the execution trace shows $n$ and $result$ being assigned its pre-loop initial value. With each repetition of the loop, result is changed, with the current value of the stepper variable $i$ added onto it. In the first repetition, $i$'s initial value is added onto result. In the next repetition, $i$ has a value of 2, so 2 is added onto result (producing 3). This continues until the last repetition, where $i$'s final value of 10 is added onto result.

We don't have to accumulate results in sums. For example, we can accumulate results in a table, or multiple tables.

**Table 13.5: Accumulating results in two tables and plotting them**

<table>
<thead>
<tr>
<th>Accumulating results in two tables and plotting them</th>
</tr>
</thead>
<tbody>
<tr>
<td>#Set parameters of script</td>
</tr>
<tr>
<td>n := 5;</td>
</tr>
<tr>
<td>#initialize accumulation variables (empty tables)</td>
</tr>
<tr>
<td>yVals := table();</td>
</tr>
<tr>
<td>tVals := table();</td>
</tr>
<tr>
<td>#repeat: calculate $10^i$ and its algorithm. Store in table</td>
</tr>
<tr>
<td>#do this for $i = 1, 2, 3, 4, 5.$</td>
</tr>
<tr>
<td>for i to n do #1 will be used for the starting value since</td>
</tr>
<tr>
<td>#there is no &quot;from&quot; given. It's optional.</td>
</tr>
<tr>
<td>tVals[i] := $10^i$;</td>
</tr>
<tr>
<td>yVals[i] := evalf(log(tVals[i]));</td>
</tr>
<tr>
<td>printf(&quot;Logarithm of %f is %f\n&quot;, tVals[i], yVals[i]);</td>
</tr>
<tr>
<td>end do;</td>
</tr>
<tr>
<td>#After the loop is finished, convert tables into a list</td>
</tr>
<tr>
<td>tList := convert(tVals, list);</td>
</tr>
<tr>
<td>yList := convert(yVals, list);</td>
</tr>
</tbody>
</table>
Do a data plot

```
plot(tList, yList, style=point, symbol=diamond, color=red, labels=['x', 'log(x)']);
```
5

\text{table}([])

\text{table}([])

10

2.302585093

Logarithm of 10.000000 is 2.302585

100

4.605170186

Logarithm of 100.000000 is 4.605170

1000

6.907755279

Logarithm of 1000.000000 is 6.907755

10000

9.210340372

Logarithm of 10000.000000 is 9.210340

100000

11.51292546

Logarithm of 100000.000000 is 11.512925

[10, 100, 1000, 10000, 100000]

[2.302585093, 4.605170186, 6.907755279, 9.210340372, 11.51292546]
13.6 Suppressing execution trace output inside loops

Note that we can make the script add the numbers from 1 to 1000 by changing the one line defining the value of $n$. However, the output grows somewhat tedious to view. We can suppress the part of the execution trace by putting a colon at the end of the "end do".

Table 13.6: Accumulation of two results in the same loop; suppression of output

| n := 5; | What the execution trace of this loop looks like without output suppression. Since the loop computes two things per repetition, and repeats 10 times, there are 10 lines of output to look at. Imagine what would happen if there $n$ were 100. |
| result := 0; | |
| result2 := 1; | |
| for i to n do | |
| result := result+i; | |
| result2 := result2*i^2; | |
| end do; | |
| printf("The sum of the numbers from 1 to %d is: %d.
", n, result); | |
| printf("The product of squares i^2 from i=1 to %d is: %d.
", n, result2); | |
The sum of the numbers from 1 to 5 is: 15.
The product of squares $i^2$ from $i=1$ to 5 is 14400.

We suppressed output from the execution trace by placing a colon at the end do, which suppresses the execution trace of everything in the loop body. This is something to do after you have gotten the loop to work correctly.

The sum of the numbers from 1 to 10 is: 55.
The product of squares $i^2$ from $i=1$ to 10 is 13168189440000.
13.7 Application of accumulative looping: making plots and movies with lists and tables

Table 13.7: Plot the position of a particle moving under Henon's rules of motion

<table>
<thead>
<tr>
<th>Code</th>
</tr>
</thead>
<tbody>
<tr>
<td>#Henon attractor</td>
</tr>
<tr>
<td>#Example of a &quot;chaotic dynamical system&quot;</td>
</tr>
<tr>
<td>#Governed by the equations of motion where</td>
</tr>
<tr>
<td>#next value of x = present value of y + 1 - 1.3*(present value of x)^2</td>
</tr>
<tr>
<td>#next value of y = 0.3*(present value of x)</td>
</tr>
<tr>
<td>#We start with x and y initially 0.</td>
</tr>
<tr>
<td>#We store successive values in a table for x and y values.</td>
</tr>
<tr>
<td>#The keys (indices) are the steps i = 0, 1, 2, 3,...</td>
</tr>
<tr>
<td>#Initialize x and y tables</td>
</tr>
<tr>
<td>x := table();</td>
</tr>
<tr>
<td>y := table();</td>
</tr>
<tr>
<td>x[0] := 0;</td>
</tr>
<tr>
<td>y[0] := 0;</td>
</tr>
<tr>
<td>#Take 5000 time steps through the system</td>
</tr>
<tr>
<td>for i from 0 to 4999 do</td>
</tr>
<tr>
<td>x[i+1] := y[i] + 1 - 1.3*(x[i])^2;</td>
</tr>
<tr>
<td>y[i+1] := .3*x[i];</td>
</tr>
<tr>
<td>end do: #Suppress output of loop  (because we've already tested it)</td>
</tr>
<tr>
<td>XL := convert(x,list);</td>
</tr>
<tr>
<td>yL := convert(y,list);</td>
</tr>
<tr>
<td>printf(&quot;Size of list :%d\n&quot;,nops(XL));</td>
</tr>
<tr>
<td>plot(xL,yL, color=red);</td>
</tr>
</tbody>
</table>
The accumulation variables are the two tables \( x \) and \( y \). As the loop repeats the tables fill with successive computed values. Even though the motion is circumscribed, the motion of the particle cannot be described by a simple mathematical formula such as the models we used for trajectories of a Human cannonball.

**Example: make a movie of the position of a particle moving under Henon's rules of motion**

```plaintext
# Henon attractor
# Example of a "chaotic dynamical system"

# Governed by the equations of motion where
# next value of \( x \) = present value of \( y \) + 1 - 1.3*(present value of \( x \))^2
# next value of \( y \) = 0.3*(present value of \( x \))

# We start with \( x \) and \( y \) initially 0.
# We store successive values in a table for \( x \) and \( y \) values.
# The keys (indices) are the steps \( i = 0, 1, 2, 3... \)

# Initialize \( x \) and \( y \) tables
x := table():
y := table():
p := table();

x[0] := 0;
y[0] := 0;
p[0] := plot([x[0]], [y[0]], style=point, color=red);

# Take 5000 time steps through the system
for i from 0 to 49 do
```

![Graph of Henon attractor](image.png)

Size of list :5001
\[ x[i+1] := y[i] + 1 - 1.3 \cdot (x[i])^2; \]
\[ y[i+1] := 0.3 \cdot x[i]; \]
\[ p[i+1] := \text{plot}([x[i+1]], [y[i+1]], \text{style=point, color=red}); \]
end do; \# Suppress output of loop (because we've already tested it)

pL := \text{convert}(p, \text{list});
printf("Size of list :\d\n", \text{nops}(pL));

with(plots):
display(pL, \text{insequence=true});

\begin{table}[h]
\begin{center}
\begin{tabular}{|l|l|l|}
\hline
 & & \\
\hline
0 & & 0 \\
\hline
\end{tabular}
\end{center}
\end{table}

Size of list :51
The accumulation variables are the two tables \( x \) and \( y \) as in the previous example, along with the table \( p \) that contains a point plot for each value of the index \( i \). At the end, instead of printing or plotting the values of \( x \) and \( y \), the \texttt{insequence} option of \texttt{display} is used to take the list of point plots and turn them into an animation. Compared to the graph, the animation is another way of gaining insight into how the particle moves over the time.

## 13.8 Troubleshooting with loops

Some of the basic problems come from forgetting the "end do" or putting too many in. Examples of this were given left out (page 190).

Another problem with a loop is that the values provided for its control cause a different effect than you expected. Discover the difference between what the rules say will happen and what you thought would happen by observing the execution trace, possibly putting in print statements.

Finally, in accumulation loops, the initial value of the accumulator variable may be wrong, or the accumulation is not happening as planned. The control of the loop may stop the accumulation too early, or too late.

### Troubleshooting loops: a script that doesn't work

```
s := 0;
x := .2;
#Approximate \( \exp(x) = 1 + x + x^2/2! + x^3/3! + \ldots \)
for i to 10 from 1 by 1 do
  #We changed the order of to and from, but that's not an error
  s := s + x^i/i!;
end do;
printf("%f compared to expected %f", s, \exp(x));
```
0.221403 compared to expected 1.221403

The execution trace seems to indicate that the right number of terms is being computed, but the answer we computed by our own programming is different than what Maple's built-in \( \exp \) function produces. The computed answer seems to be one less than the answer we expect. This is different than the other examples previously shown where the programmed results agreed quite closely with Maple's built-in computation.

Rerunning the script with more information generated by the execution trace

```plaintext
s := 0;
x := .2;
#Approximate \( \exp(x) = 1 + x + x^2/2! + x^3/3! + \ldots \)
for i to 10 from 1 by 1 do
    printf("i= %d, x = %f, s = %f, x^i/i! = %f\n", i, x, s, x^i/i!);
s := s + x^i/i!;
end do;
printf("%f compared to expected %f", s, exp(x));
```

We enhance the execution trace by having it print out the values of all the variables with each repetition. It's work to enter in this printf statement, but it pays off in helping us find the mistake more quickly. Comparing the actual value of \( s \) with what the program comment tells us to expect term by term, we see that while \( s \) starts off at zero in the first iteration, the first term \( \left( \frac{x}{1!} \right) \) is .2, not 1 as we expect. Looking at the second and third terms, we realize that we have forgotten to add in the first term to \( s \).

There are two ways to do this -- either to start at \( i=0 \) (since \( \frac{x}{0!} \) is 1) or to keep the loop as it is and just initialize \( s \) to 1 instead of zero before the loop starts. We opt for the latter since it doesn't require remembering facts from high school days such as that \( 0!=1 \).

Rerunning the script with repairs

```plaintext
s := 1;
x := .2;
```
Approximate \( \exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \)

```python
for i to 10 from 1 by 1 do
    printf("i= %d, x = %f, s = %f, x^i/i! = %f\n", i, x, s, x^i/i!);
    s := s + x^i/i!;
end do;
printf("%f compared to expected %f", s, exp(x));
```

```
1
0.2
i= 1, x = 0.200000, s = 1.000000, x^i/i! = 0.200000
1.2
i= 2, x = 0.200000, s = 1.200000, x^i/i! = 0.020000
1.22000000
i= 3, x = 0.200000, s = 1.220000, x^i/i! = 0.001333
1.22133333
i= 4, x = 0.200000, s = 1.221333, x^i/i! = 0.000067
1.22140000
i= 5, x = 0.200000, s = 1.221400, x^i/i! = 0.000003
1.22140266
i= 6, x = 0.200000, s = 1.221403, x^i/i! = 0.000000
1.22140275
i= 7, x = 0.200000, s = 1.221403, x^i/i! = 0.000000
1.22140275
i= 8, x = 0.200000, s = 1.221403, x^i/i! = 0.000000
1.22140275
i= 9, x = 0.200000, s = 1.221403, x^i/i! = 0.000000
1.22140275
i= 10, x = 0.200000, s = 1.221403, x^i/i! = 0.000000
1.22140275
1.221403 compared to expected 1.221403
```

Making this change produces the expected agreement. At this point, we would "comment out" the printf that we put in to enhance the execution trace, and could change the value of \( x \) to compute an approximation to \( \exp(x) \) for other values of \( x \).

More pernicious are loops which do not give error messages but either stop with the wrong answer, or do not stop.

Runaway loops can be stopped by hitting the red "stop hand". But the result is not really satisfactory, as illustrated in the example below.
A runaway "infinite" loop

<table>
<thead>
<tr>
<th>for x from 3.0 by .1 do</th>
</tr>
</thead>
<tbody>
<tr>
<td>print(&quot;sin(x)&quot;, sin(x));</td>
</tr>
<tr>
<td>end do;</td>
</tr>
</tbody>
</table>

"sin(x)", 3.0, 0.1411200081
"sin(x)", 3.1, 0.04158066243
"sin(x)", 3.2, -0.05837414343
"sin(x)", 3.3, -0.1577456941
"sin(x)", 3.4, -0.2555411020
"sin(x)", 3.5, -0.3507832277
"sin(x)", 3.6, -0.4425204433
"sin(x)", 3.7, -0.5298361409
etc., etc.

The problem with this loop is that it will never stop. Because we mistakenly left off the "to finalValue" portion of the loop (which is legal, even if dangerous), the loop will just keep repeating, adding .1 onto the value of x.

While in principle the "red stop hand" is useful for computations that don't do a lot of printing, in practice it is **bad news** if you make a mistake and execute this kind of infinite loop. Like the Sorcerer's Apprentice (see <http://en.wikipedia.org/wiki/The_Sorcerer%27s_Apprentice> if you don't know the cultural reference), you will see seemingly endless lines of output. Clicking on the red stop hand will eventually stop the output, but not until after a long, long wait while all the output that resulted from before you used the hand scrolls by. This is because Maple typically generates thousands of repetitions of the loop in less than a second. It takes a long time for it to print all the output from the repetitions, and you won't be able shut off the printing with the red hand.

About the only way to make the output stop is to force quit out of the Maple application. This will cause you to lose the current state of the worksheet!

If you've saved your worksheet while you were working, then you can recover something close to the state before the "infinite loop" made a mockery of your session. Alternatively, you can recover the worksheet using File->Recent Documents -> Restore this section (page 20) as described in this section (page 20).
### 13.9 Summary of Chapter 13

<table>
<thead>
<tr>
<th>Repetition type</th>
<th>Example</th>
</tr>
</thead>
</table>
| Repetition with `to ... do ... end do` | to 5 do
   - printf("I will not burp in class.
   end do;
   printf("The repetition is over.\n"); |
| Repetition with `for ... to ... do ... end do` | #Set parameters
   - n := 5;
   
   #Repeat an action with i stepping through the values
   #1, 2, 3, 4, ... n
   - for i to n do
     - printf("Repetition %d: sin(%d) = %f.\n", i, i,
      - evalf(sin(i)))
   od;
   printf("The repetition is over.\n"); |
| Repetition with `for ... to ... by ... do ... end do` | #Final value parameter
   - fv := 1.2;
   
   #Repeat an action with i stepping through the values
   #0, .1, .2, .3, ....
   - for x from 0 by .1 to fv do
     - printf("sin(%f) = %f.\n", x, evalf(sin(x)))
   end do;
   printf("The repetition is over.\n"); |
| Repetition (incrementing) with `for ... from ... to ... by ... do ... end do` | for x from 0 by .1 to 1 do
   - printf("sin(%f) = %f.\n", x, evalf(sin(x)));
   end do; |
| Repetition (decrementing) with `for ... to ... do ... end do` | for x by -.3 to -1 do
   - printf("sqrt(%f) = %a\n", x, evalf(sqrt(x)));
   end do; |

<table>
<thead>
<tr>
<th>Accumulation loop type</th>
<th>Example</th>
</tr>
</thead>
</table>
| Prints result          | #Set parameters of script
   - n := 10;
   
   #initialize accumulation variable
   - result := 0;
   
   #repeat: add i to the accumulated sum
   - for i to n do
     - result := result+i;
   end do;
   printf("The sum of the numbers from 1 to %d is: %d", |

---

**Example Repetition type**

- **Repetition with `to ... do ... end do`**
  ```
  to 5 do
    printf("I will not burp in class.\n");
  end do;
  printf("The repetition is over.\n");
  ```

- **Repetition with `for ... to ... do ... end do`**
  ```
  #Set parameters
  n := 5;
  
  #Repeat an action with i stepping through the values
  #1, 2, 3, 4, ... n
  for i to n do
    printf("Repetition %d: sin(%d) = %f.\n", i, i,
    evalf(sin(i)))
  od;
  printf("The repetition is over.\n");
  ```

- **Repetition with `for ... to ... by ... do ... end do`**
  ```
  #Final value parameter
  fv := 1.2;
  
  #Repeat an action with i stepping through the values
  #0, .1, .2, .3, ....
  for x from 0 by .1 to fv do
    printf("sin(%f) = %f.\n", x, evalf(sin(x)))
  end do;
  printf("The repetition is over.\n");
  ```

- **Repetition (incrementing) with `for ... from ... to ... by ... do ... end do`**
  ```
  for x from 0 by .1 to 1 do
    printf("sin(%f) = %f.\n", x, evalf(sin(x)));
  end do;
  ```

- **Repetition (decrementing) with `for ... to ... do ... end do`**
  ```
  for x by -.3 to -1 do
    printf("sqrt(%f) = %a\n", x, evalf(sqrt(x)));
  end do;
  ```

- **Accumulation loop type**
  ```
  #Set parameters of script
  n := 10;
  
  #initialize accumulation variable
  result := 0;
  
  #repeat: add i to the accumulated sum
  for i to n do
    result := result+i;
  end do;
  printf("The sum of the numbers from 1 to %d is: %d", |
  ```
n := 5;

#Set parameters of script
n := 5;

#Initialize accumulation variables (empty tables)
yVals := table();
tVals := table();

#Repeat: calculate 10^i and its algorithm. Store
#in table
#Do this for i = 1, 2, 3, 4, 5.

for i to n do #1 will be used for the starting value
  #There is no "from" given. It's optional.
  tVals[i] := 10^i;
  yVals[i] := evalf(log(tVals[i]));
  printf("Logarithm of %f is %f\n",tVals[i],yVals[i]);
end do;

#After the loop is finished, convert tables into a
list

for i to n do #1 will be used for the starting value
  #There is no "from" given. It's optional.
  tVals[i] := 10^i;
  yVals[i] := evalf(log(tVals[i]));
  printf("Logarithm of %f is %f\n",tVals[i],yVals[i]);
end do;

#Data plot

plot(tList,yList, style=point, symbol=diamond, color=red, labels=["x","log(x)"]);

n := 5;
result := 0;
result2 := 1;
for i to n do
  result := result+i;
  result2 := result2*i^2;
end do;

printf("The sum of the numbers from 1 to %d is:
%d\n", n, result);

printf("The product of squares i^2 from i=1 to %d is
%d\n",n,result2);

s := 1;
x := .2;

#Approximate exp(x) = 1 + x + x^2/2! + x^3/3! + ...

for i to 10 from 1 by 1 do
  printf("i= %d, x = %f, s = %f, x^i/i! = %f\n", i, x, s, x^i/i!);
  s := s + x^i/i!;
end do;
printf("%f compared to expected %f", s, exp(x));

<table>
<thead>
<tr>
<th>Do not create an infinite loop</th>
</tr>
</thead>
<tbody>
<tr>
<td># Do not run this</td>
</tr>
<tr>
<td># We do not have a &quot;to finalValue&quot; portion in our for loop</td>
</tr>
<tr>
<td>for x from 3.0 by .1 do</td>
</tr>
<tr>
<td>print(&quot;sin(x)&quot;, sin(x));</td>
</tr>
<tr>
<td>end do;</td>
</tr>
</tbody>
</table>
14 More on repetition and looping

14.1 Chapter Overview

In the previous chapter we have seen how for...do...end do can be used to repeat a segment of code, possibly with a stepping variable that is automatically managed by Maple during the execution. In this chapter we look at other ways of getting repetition to occur: through some of the built-in shortcuts such as seq, sum, and addition of lists.

We also look at while .. do ... end do which is a way of having repetition occur while a condition holds.

for ... while ... do .... end do is a way of having both a stepping variable with an upper limit, and an additional condition that can end the repetition early if it becomes false.

14.2 Implicit looping: seq, list addition, sum

In chapter 11, we learned about for and its ability to loop with an index (stepping) variable. While this is a marvelous abbreviation, it still takes work to set up. Certain repetitious actions occur so commonly that many programming languages provided even shorter ways of performing looping idioms.

For example, suppose we wanted to build a list consisting of the numbers 1,4,9, 25, ... 169. We could type such a list in, but from what we know of looping we suspect that there's an easier way. Using the ideas we've seen so far, we could use a table valueTab that has an accumulation variable L. With i as the stepping variable going from 1 to 100, we would fill up the table with \( i^2 \) values. After the loop finishes, we could convert the table into a list to get our result:

Table 14.1: Making a list through a loop

<table>
<thead>
<tr>
<th>Making a list through a loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>#initialize table</td>
</tr>
<tr>
<td>valueTab := table();</td>
</tr>
<tr>
<td>#fill up the table</td>
</tr>
<tr>
<td>for i from 1 to 13 do</td>
</tr>
<tr>
<td>valueTab[i] := i^2;</td>
</tr>
<tr>
<td>end do;</td>
</tr>
<tr>
<td>#Produce list as result</td>
</tr>
<tr>
<td>L := convert(valueTab,list);</td>
</tr>
<tr>
<td>printf(&quot;L is: %a&quot;, L);</td>
</tr>
</tbody>
</table>
The nice thing about this loop is that it is no more work for us to create a list with 100 items 1, 4, 9, ... 10,000. All we have to do is change the "13".

However, there is an even shorter way to create this list. Maple has a function seq whose syntax borrows heavily from for. The seq function creates a sequence as indicated by the values of the stepping variable. To get a list, we convert the sequence into a list, as explained in Chapter 12 (page 185).

Table 14.2: Making a list with seq

Making a list with seq

<table>
<thead>
<tr>
<th>#initialize table</th>
</tr>
</thead>
<tbody>
<tr>
<td>#Create a sequence</td>
</tr>
<tr>
<td>s := seq(i^2,i=1..13);</td>
</tr>
<tr>
<td>#Convert it to a list</td>
</tr>
<tr>
<td>L := [s];</td>
</tr>
<tr>
<td>printf(&quot;L is: %a&quot;, L);</td>
</tr>
</tbody>
</table>

L is: [1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169]

With this kind of facility, it is easy to quickly solve many problems that are solved with Maple functions that take lists.

Table 14.3: Creating a list of values with seq

Creating a list of values with seq

Create a list of the approximations to $\sqrt{1}$, $\sqrt{2}$, ... $\sqrt{20}$. 
Creating a list of values with `seq`

```latex
\begin{verbatim}
[seq(evalf(sqrt(i)), i = 1..20)]
    [1., 1.414213562, 1.732050808, 2., 2.236067797, 2.449489743, 2.645751311, 2.828427124, 3.,
     3.16227766, 3.31662479, 3.464101616, 3.605551275, 3.741657387, 3.872983346, 4.,
     4.123105626, 4.242640665, 4.35898984, 4.472135954]
\end{verbatim}
```

The general forms of `seq` are

\[ \text{seq( expression involving steppingVariable, steppingVariable=range) \ and } \]
\[ \text{seq( expression involving steppingVariable, steppingVariable=range, stepValue).} \]

The second form behaves like the `for...from...to...by...do...end do` described in the previous chapter.

### Table 14.4: Giving lists to Maple functions using `seq` with a step value

<table>
<thead>
<tr>
<th>Giving lists to Maple functions using <code>seq</code> with a step value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Create a list of the approximations to ( \sqrt{2} ), ( 2.1 ), ( \ldots ), ( 5.0 ).</td>
</tr>
</tbody>
</table>
| \begin{verbatim}
[seq(evalf(sqrt(i)), i = 2..5, 1)]
    [1.414213562, 1.449317675, 1.483236977, 1.516575089, 1.549193338, 1.581138830, 1.612451550,
     1.643167673, 1.673320053, 1.702938637, 1.732050808, 1.760681686, 1.788854382, 1.816590212,
     1.84390891, 1.870828693, 1.89736659, 1.923538406, 1.949358889, 1.974841766, 2.000000000,
     2.022845673, 2.049390153, 2.07644135, 2.097961796, 2.121320344, 2.144761059, 2.16794339,
     2.190890230, 2.213594362, 2.236067977]
\end{verbatim} |

Adding together the elements of two lists is another common operation that has a built-in shortcut.

### Table 14.5: Adding together two lists with +

<table>
<thead>
<tr>
<th>Adding together two lists with +</th>
</tr>
</thead>
<tbody>
<tr>
<td>Create a list of values of ( \sin\left(\frac{n \pi}{10}\right) ) (approximately).</td>
</tr>
<tr>
<td>( L\sin := \text{evalf}\left(\left[\text{seq}\left(\sin\left(\frac{i \cdot \pi}{10}\right), i = 1..10\right)\right]\right) )</td>
</tr>
</tbody>
</table>
| \begin{verbatim}
[0.3090169944, 0.5877852524, 0.8090169944, 0.9510565165, 1., 0.9510565165, 0.8090169944,
     0.5877852524, 0.3090169944, 0.]
\end{verbatim} |
| Create a similar list of values for \( \cos\left(\frac{n \pi}{10}\right) \) |
| \( L\cos := \text{evalf}\left(\left[\text{seq}\left(\cos\left(\frac{i \cdot \pi}{10}\right), i = 1..10\right)\right]\right) \) |
| \begin{verbatim}
[0.9510565163, 0.8090169943, 0.5877852522, 0.3090169938, 0., -0.3090169938, -0.5877852522,
     -0.8090169943, -0.9510565163, -1.]
\end{verbatim} |
| Add them together using "+". |
| \( L\text{sum} := L\sin + L\cos \) |
| \begin{verbatim}
[1.260073511, 1.396802247, 1.396802247, 1.260073510, 1., 0.6420395227, 0.2212317422,
     -0.2212317419, -0.6420395219, -1.]
\end{verbatim} |
Adding together two lists with +

We don't get component-wise "automatic multiplication" with "*". Maple doesn't give an error, but it doesn't do the multiplication, either.

\[ L_{\text{wrong}} := \text{Lsin-Lcos} \]

\[
\begin{bmatrix}
0.3090169944, 0.5877852524, 0.8090169944, 0.9510565165, 1., 0.9510565165, 0.8090169944, \\
0.5877852524, 0.3090169944, 0. \end{bmatrix} \]

We get this done without writing a loop to do it with the use of seq.

\[ L_{\text{right}} := \text{seq}(\text{Lsin}[i]-\text{Lcos}[i], i = 1..\text{numelems}(\text{Lsin})) \]

\[
\begin{bmatrix}
0.2938926262, 0.4755282582, 0.4755282581, 0.2938926257, 0., -0.2938826257, -0.4755282581, \\
-0.4755282582, -0.2938926262, -0. \end{bmatrix} \]

"Automatic addition" works with numbers, expressions, lists, sequences, ranges, and equations, but does not work with tables, plots, or sets. "Automatic multiplication" only works with numbers and expressions. Maple does have additional data containers with use with linear algebra, vectors, and matrices for which both addition and multiplication work -- but that should wait until you need to do "linear algebra".

### 14.3 More implicit looping with sum

Maple has another function which will use looping when it is expedient: summation. This is available textually through the \texttt{sum} operation, and through the Expression Palette as \[ \sum_{i=k}^{n} f. \] The general form of \texttt{sum} is

\[ \texttt{sum(expression, indexVariable=range)}. \] When the range is numerical and contains a manageable number of values, Maple will automatically do the summation through a loop.

\begin{table}[h]
\centering
\begin{tabular}{|c|}
\hline
| Automatic summation |
\hline

\[ \sum_{i=1}^{100} i! \] #This does a summation using the Expression Palette

\[ 942690016837099792608598341244735398720707226139826724429383593056246782234795060234: \\
002940935991364669860912434743264762282687003820556442336529820420940313 \]

\[ \texttt{sum(sqrt(i), i = 1..10)} \] #This does summation using the textual form of the operation

\[ 1 + \sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{8} + \sqrt{9} + \sqrt{10} \]

\text{at 5 digits}

\[ 22.468 \]  

\hline
\end{tabular}
\end{table}

However, there is more to the \texttt{sum} function than just an abbreviation of a \texttt{for} loop. One can give non-numeric value as part of the range, and Maple will use mathematical analysis to give a formula for the summation if it can. Furthermore, it will this analysis to provide a short-cut to the looping if it turns out to be faster.

Table 14.7: sum and non-looping behavior

<table>
<thead>
<tr>
<th>sum and non-looping behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sum_{i=1}^{n} i^4$ Maple knows the formula for the sum of 4th powers of integers.</td>
</tr>
<tr>
<td>$\frac{1}{5} (n + 1)^5 - \frac{1}{2} (n + 1)^4 + \frac{1}{3} (n + 1)^3 - \frac{1}{30} n - \frac{1}{30}$ (14.12)</td>
</tr>
</tbody>
</table>

If we did the following with a for loop, it would take literally centuries for even a supercomputer to complete. Rather than do that, Maple figures out the formula for the sum of squares and plugs in $n = 10^{30}$.

$\sum (i^2, i = 1..10^{30})$

One annoying aspect of using sum is that it expects the index variable to be unassigned (have no value). If you discover that you've used the index variable previously in the session, then you have to unassign it. This is similar to the situation that arose when solve broke because the variable being solved for had been previously assigned a value, or the variable was already the name of a built-in function.

<table>
<thead>
<tr>
<th>sum and previously assigned values of the index variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>restart;</td>
</tr>
<tr>
<td>L := [3.6, 3.9, 2.1, 3.9, 2.1, 0.7, 1.3, 2.8, 3.6, 3.7, 2.9];</td>
</tr>
<tr>
<td>total := sum(L[i], i=1..nops(L));</td>
</tr>
<tr>
<td>avg := total/nops(L);</td>
</tr>
<tr>
<td>for i from 1 to 10 do</td>
</tr>
<tr>
<td>printf(&quot;Grade = %f, deviation= %f \n&quot;, L[i], L[i]-avg);</td>
</tr>
<tr>
<td>end do;</td>
</tr>
<tr>
<td>stdev := sum((L[i]-avg)^2, i=1..nops(L))/avg; # this is broken</td>
</tr>
<tr>
<td>i := 'i'; # unassign the index variable in the sum.</td>
</tr>
<tr>
<td>stdev := sum((L[i]-avg)^2, i=1..nops(L))/avg; # this works</td>
</tr>
</tbody>
</table>

[3.6, 3.9, 2.1, 3.9, 2.1, 0.7, 1.3, 2.8, 3.6, 3.7, 2.9]

30.6

2.781818182

Grade = 3.600000.2, deviation= 0.818182.2
Grade = 3.900000.2, deviation= 1.118182.2
Grade = 2.100000.2, deviation= -0.681818.2
Grade = 3.900000.2, deviation= 1.118182.2
Grade = 2.100000.2, deviation= -0.681818.2
Grade = 0.700000.2, deviation= -2.081818.2
Grade = 1.300000.2, deviation= -1.481818.2
Grade = 2.800000.2, deviation= 0.018182.2
Grade = 3.600000.2, deviation= 0.818182.2
Grade = 3.700000.2, deviation= 0.918182.2

Error, (in sum) summation variable previously assigned, second argument evaluates to 11 = 1 .. 11

i

4.369934642 (14.14)
The error message comes when we try to compute the standard deviation. While the use of \( i \) in the computation of total had no problems, the use of \( i \) in the computation of the standard deviation occurred after it had been assigned values during the for loop. The way to fix this is to either use another variable that is known not to have been assigned a value (e.g. \( k \)), or to unassign \( i \) before using it in the second sum.

### 14.4 Conditional repetition through while... do... end do

**for ... do ... end do** allows you to repeat a segment of code under control of an index variable. Typically, the starting, ending and stepping values of the index variable are set at the start of execution of the for. This means that the number of times that the loop is repeated is known in advance of the execution.

There are other situations where the number of repetitions is not known in advance. The programmer wants "keep on repeating this until a condition is reached", or alternatively "keep on repeating this while a condition is true". Maple has a way of doing this, with syntax similar to for.

#### Table 14.8: The general form of the while statement

<table>
<thead>
<tr>
<th>The general form of the while statement</th>
</tr>
</thead>
<tbody>
<tr>
<td>while condition do</td>
</tr>
<tr>
<td>loop body;</td>
</tr>
<tr>
<td>end do;</td>
</tr>
</tbody>
</table>

The condition is evaluated only once per repetition, at the start before any of the code in the loop body is executed. If the condition is true, then the entire loop body is executed. If not, it decides that the repetition is finished and moves onto the next instruction after the end do.

The condition is evaluated at the start of the loop iteration. It does not matter if the condition changes in the middle of the loop body; this won't terminate execution of the loop until the execution reaches the end of the loop body and the condition looked at again before the start of the next repetition of the loop.

The continuation condition is usually an inequality such as \( x<.1 \) or \( \text{abs}(y-z)>\text{tol} \). It always must be an expression that can be evaluated to be true or false. If the inequality is numeric such as \( x<.1 \), variables in the condition such as \( x \) must have been assigned a numeric value earlier in the Maple session.

#### Table 14.9: Repeating while a condition is true

**Example: Repeating while a condition is true: accumulate terms of a sum until next term is very small.**

The code region below computes the sum

\[
x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots
\]

until it finds a term that is tiny (less than the value of the parameter \( \text{tol} \)). It does this through repeated addition to the accumulation variable \( s \). The continuation condition repeats the loop body while the value of \( \frac{x^i}{i!} > \text{tol} \). This condition is evaluated only once per repetition, just before all the code in the loop body is executed. The particular term calculated during a repetition is \( \left( \frac{(i-1)}{2} \right) \frac{x^i}{i!} \) for \( i = 1, 3, 5, \ldots \).

Some of you who have taken calculus may recognize this sum as an approximation to \( \sin(x) \). Floating point calculations in Maple, Matlab, or calculators such as the TI-89 use sums or similar means of approximations to provide values for \( \sin \) and other mathematical functions.

```maple
# Initialize script parameters
# tol is the size of the stopping criterion.
tol := 10e-7;
# x is the value of sin(x) that we are trying to approximate
x := .3;
```

---

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# Initialize accumulation variables
i := 1;  # i will be 1, 3, 5, 7, 9 as the repetition continues
term := x;
s := term;  # s accumulates the sum, we initialize it to the first term.

# Add on successive terms
while abs(term) >= tol do
    i := i+2;
term := (-1)^((i-1)/2)*(x^i)/i!;
s := s+term;
printf("Term is %e, sum is %f.\n", term,s);
end do:  # Suppress execution trace inside the loop (but printf still work).

# Print some summary messages.
printf("Sum is: %e compared to %e.\n",s, sin(x));
printf("Stopped when i=%d, term=%e.\n", i, term);

Term is -4.500000e-03, sum is 0.295500.
Term is 2.025000e-05, sum is 0.295520.
Term is -4.339286e-08, sum is 0.295520.
Sum is: 2.955202e-01 compared to 2.955202e-01.
Stopped when i=7, term=-4.339286e-08.

Notes
The terms being computed are sometimes positive and sometimes negative. This is why the continuation condition uses the absolute value of the term rather than just the value of the term. It says "do another repetition while the term is bigger than \( tol \). The last iteration of the loop computes a term that's \(-4.3392857140^8\). After that, the condition is not true, so the repetition stops. The final result agrees to ten decimal places with Maple's built-in \( \sin \) function's.

The textual form of inequalities such as "<" and ">" are easy since there is a symbol on the keyboard for them. The following table describes the textual version of inequalities such as \( \leq \) for which there is no single symbol on the keyboard. The symbols for inequalities are sometimes referred to as relational operators.

### Table 14.10: Textual entry of relational operators

<table>
<thead>
<tr>
<th>Mathematical operator</th>
<th>Name</th>
<th>Textual entry</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;</td>
<td>Less than</td>
<td>&lt;</td>
</tr>
<tr>
<td>&gt;</td>
<td>Greater than</td>
<td>&gt;</td>
</tr>
<tr>
<td>=</td>
<td>Equals</td>
<td>=</td>
</tr>
<tr>
<td>≠</td>
<td>Not equals</td>
<td>&lt;&gt;</td>
</tr>
<tr>
<td>≤</td>
<td>Less than or equals</td>
<td>&lt;=</td>
</tr>
</tbody>
</table>
Another example of accumulation

This example is similar to the previous one. It computes successive terms of a sum for the natural logarithm:

$$\ln(x) = x - 1 - \frac{1}{2} (x - 1)^2 + \frac{1}{3} (x - 1)^3 - \frac{1}{4} (x - 1)^4 + \frac{1}{5} (x - 1)^5 + \ldots$$

but stops when the term becomes smaller than the tolerance.

```plaintext
#Computation of logarithms through sums.
# Initialize script parameters
# tol is the size of the stopping criterion.
tol := 1e-4;
# x is the value of sin(x) that we are trying to approximate
x := .2;
xm1 := x-1.0: #"x minus 1", since we use it a lot

# Initialize accumulation variables
i := 1; # i will be 1, 2, 3, 4, 5, .. as the repetition continues
term := xm1;
s := term; # s accumulates the sum, we initialize it to the first term.
xm1power := xm1: # to be used for powers of x minus 1, initially just xm1

printf("x is %f, tol is %e\n", x, tol);
# print out all values on one line as a check before we start.

# add on successive terms
while abs(term)>tol do
  i := i+1; # compute next value of the denominator
  xm1power := -xm1power*xm1; # Next power of x-1, adjusted by +- sign.
term := xm1power/i; # Next term of the sum
  s := s+term; # Add term to sum.
  printf("term is %e, sum is %f\n", term, s);
end do:

printf("sum is: %e compared to: %e\n", s, ln(x));
print(1, "terms used");
```
The approximation does not produce ten decimal places of agreement with the logarithm as calculated by the Maple library, but it's closing in on that when the repetition stops. This is because \( \text{tol} \) is set to \( 10^{-4} \), so the repetition stops as soon as a \( \text{term} \) is computed that is less than that in absolute value. If we really wanted ten places of accuracy, we might want to set \( \text{tol} \) to \( 10^{-10} \). It would make the repetition go on longer, but would produce more decimal places of agreement.

It would be naive to presume that just because we set \( \text{tol} \) to \( 10^{-10} \) that the result will agree with the correct result to 10 decimal places. (What actually happens? Try it.) Mathematical analysis can be used to provide guarantees or proofs that things will work out. The area of mathematics that studies this is called numerical analysis. Many undergraduate mathematics or computer science students interested in devising efficient accurate programs for numerical computation take a numerical analysis course as a junior or senior.
14.5 Patterns of looping

We have introduced quite a lot of machinery. It can be easy for beginners to be uncertain about the connection between *intent* (what you want to achieve) and *syntax* (how you say it in the programming language). Here as a short table summarizing the connections we have introduced so far.

<table>
<thead>
<tr>
<th>Patterns of looping</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Intent</strong></td>
<td><strong>Syntax</strong></td>
</tr>
<tr>
<td>Repeat exactly the same thing <em>n</em> times</td>
<td>to 5 do</td>
</tr>
<tr>
<td></td>
<td>[ \text{print}(&quot;I will not teach others to fly&quot;); ]</td>
</tr>
<tr>
<td></td>
<td>end do;</td>
</tr>
<tr>
<td></td>
<td>&quot;I will not teach others to fly&quot;</td>
</tr>
<tr>
<td></td>
<td>&quot;I will not teach others to fly&quot;</td>
</tr>
<tr>
<td></td>
<td>&quot;I will not teach others to fly&quot;</td>
</tr>
<tr>
<td></td>
<td>&quot;I will not teach others to fly&quot;</td>
</tr>
<tr>
<td></td>
<td>&quot;I will not teach others to fly&quot;</td>
</tr>
<tr>
<td></td>
<td>(14.17)</td>
</tr>
<tr>
<td>The action within the loop depends on the value of variable stepping from an initial to a final value, adding a fixed amount to the stepper variable between each step</td>
<td>for t from 0 by .1 to 1 do</td>
</tr>
<tr>
<td></td>
<td>[ \text{printf}(\text{&quot;arccos(%f) = %f\n&quot;}, t, \text{arccos(t))} ]</td>
</tr>
<tr>
<td></td>
<td>end do;</td>
</tr>
</tbody>
</table>
| | \begin{align*}
| & \text{arccos}(0.000000) = 1.570796 \\
| & \text{arccos}(0.100000) = 1.470629 \\
| & \text{arccos}(0.200000) = 1.369438 \\
| & \text{arccos}(0.300000) = 1.266104 \\
| & \text{arccos}(0.400000) = 1.159279 \\
| & \text{arccos}(0.500000) = 1.047198 \\
| & \text{arccos}(0.600000) = 0.927295 \\
| & \text{arccos}(0.700000) = 0.795399 \\
| & \text{arccos}(0.800000) = 0.643501 \\
| & \text{arccos}(0.900000) = 0.451027 \\
| & \text{arccos}(1.000000) = 0.000000 \\
| \end{align*} |
| | (14.18) |
| Accumulation: Variables within the loop accumulate results. Each repetition incrementally changes onto the partial result. When the repetition is finished, the entire result has been computed. | x := .3: \#Parameter |
| | prod := 1: \#Accumulation variable |
| | \#"sum" and "product" are reserved Maple names \#So we should use something else like "s" or "prod" \#for variables. |
| | \#Compute the product of \((x+2)\cdot(x+3)\cdot(x+4)\cdot\ldots\cdot(x+15)\) |
| | for i by 2 to 15 do |
| | \[ \text{prod} := (x+i)\cdot\text{prod}; \] |
| | end do: \#Suppress execution trace inside the loop |
| | \text{print}(\text{prod}); |
| | 3.549445283 \times 10^6 |
| | (14.18) |
This computes an approximation to

\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots. \]

# Initialize script parameters
\[ x := .2; \]
\[ tol := 10^{-6}; \]

# Accumulation variables, initialized
\[ i := 1; \]
\[ s := 1; \quad \# \text{First term of sum} \]
\[ t := 1; \quad \# \text{Next term of sum} \]

while (abs(t) >= tol) do
    \[ t := \frac{x^i}{i!}; \]
    \[ s := s + t; \]
    \[ i := i + 1; \]
end do;

printf("compare %e to %e\n", s, exp(x));

compare 1.221403e+00 to 1.221403e+00

14.6 Combining for and while

Maple's syntax allows the use of a stepper variable and a continuation condition (other than that of the stepper variable reaching a final value.)

\[ \text{for stepperVariable from initialValue to finalValue by stepValue while condition do .... end do} \]

promotes a repetition which stops when either the stepperVariable reaches the finalValue, or the condition becomes false. If the to initialValue is omitted, then the while condition is the sole criterion used to decide whether to continue the repetition. This can abbreviate the coding in many standard cases.

Table 14.13: Example of for and while repetition

Example of for and while repetition: accumulate terms of a sum until next term is very small.

The code region below computes the sum

\[ x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \]

until it finds a term that is tiny (less than the value of the parameter tol). Compared to the previousExample (page 214), the main difference is that we can skip the initialization and increment of the stepper variable i since the for loop handles it for us automatically. Since there is no finalValue, the loop will not stop because the stepper variable reached a final value. However, at the start of each repetition, the condition abs(term) >= tol will also be checked. This is what stops the repetition after a few trips through the loop.

# Initialize script parameters
# tol is the size of the stopping criterion.
\[ \text{tol} := 10^{-7}; \]
# x is the value of sin(x) that we are trying to approximate
\[ x := .3; \]

# Initialize accumulation variables
term := x;
s := term; # s accumulates the sum, we initialize it to the first term.

# add on successive terms
for i from 1 by 2 while abs(term)>={tol} do
  i := i+2;
  term := (-1)^((i-1)/2)*(x^i)/i!;
  printf("term is %e\n", term);
  s := s+term;
end do;

# Print out the final sum, and compare it to what it's supposed to be.
printf("sum is: %e compared to %e.\n", s, sin(x));

0.0000010
0.3
0.3
0.3
3
-0.00450000000

term is -4.500000e-03
0.295500000
7
-4.339285714e-8

term is -4.339286e-08
0.2954999566

sum is: 2.955000e-01 compared to 2.955202e-01.

We can get more decimal places of agreement between our approximation and the reference result by changing the value of {tol}. This will cause more repetitions of the loop.

This kind of repetition is often used in practice in situations where there is no built-in function that will compute the result for you. You'd want to ask a numerical analyst to provide assurances that it's going to work well.
14.7 Troubleshooting while loops: loops that don’t stop

We’ve already reviewed some of the basic problems with loops that come from forgetting to put in an `end do` or `do`, or putting too many in. These typically give some kind of Error, message and were discussed left out (page 190).

More pernicious are loops which do not give error messages but either stop with the wrong answer, or do not stop.

You can tell that this is happening because you are either flooded with a lot of output, or the status window in the lower left hand corner of the worksheet says Evaluating…. but never seems to return to Ready.

Table 14.14: A runaway infinite loop

<table>
<thead>
<tr>
<th>A runaway &quot;infinite&quot; loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>x := 3.0;</td>
</tr>
<tr>
<td>while (x&lt;10) do</td>
</tr>
<tr>
<td>print(&quot;sin(x)&quot;, sin(x));</td>
</tr>
<tr>
<td>end do;</td>
</tr>
<tr>
<td>3.0</td>
</tr>
<tr>
<td>&quot;sin(x)&quot;, 0.1411200081</td>
</tr>
<tr>
<td>&quot;sin(x)&quot;, 0.1411200081</td>
</tr>
<tr>
<td>&quot;sin(x)&quot;, 0.1411200081</td>
</tr>
<tr>
<td>&quot;sin(x)&quot;, 0.1411200081</td>
</tr>
<tr>
<td>etc., etc.</td>
</tr>
</tbody>
</table>

The problem with this loop is that it will never stop. The continuation condition is x<10, which is true initially. Because the loop body never changes the value of x, it will continue to be true for every successive repetition. Presumably the programmer made a mistake and forgot to put in lines of code into the loop body that changed the value of x and (with enough repetition) made it eventually grow to 10 or larger.

In theory, you can stop long-running computations such as infinite loops with the "red stop hand" found on the Maple tool bar. Clicking on the red stop hand will eventually stop the output, but not until after a long, long wait while all the output that resulted before you told the computer to stop looping, is printed. It typically takes a long time for it to print all the output from the repetitions, and you won't be able shut off the printing with the red hand.

In practice, if you have a runaway loop you often will have to force-quit your Maple session and recover the most recent version of your worksheet through File -> Recent Backups-> Recover Backup, as described in chapter 2. If you absolutely must execute a runaway loop again, first change it to print very little so that it will stop before you get old waiting for it.
Interrupting an evaluation with the "red stop hand"

After entering an equation to solve, we have realized that we really don't want to see millions of roots listed, assuming that Maple can find them all. So we click on the red hand (circled in green).

After a significant pause, the message "computation interrupted" appears and we can enter another computation for Maple to perform. There is no result from the interrupted computation, so no result is displayed. The hand becomes gray instead of red when execution is over.

This technique does not work well if the long running computation prints a lot, as might occur in a runaway loop with a verbose execution trace.

14.8 Troubleshooting loops: loops that give the wrong answer

Loops that return the wrong answer need to be handled analytically -- you need to figure out what should be happening and why. Then compare what should be happening with what the execution trace is saying does happen to diagnose your problem. "Sick loops" take a fair bit of thought to work through, since you are looking a code that gets repeatedly executed, possibly many times.

Some tips for handling loops include:

a) Develop incrementally. Set your loops up for a few repetitions (two or three) initially. Verify that things work before you try to make the repetitions go on longer.

b) Use print/printf statements within the loop to make it clearer what is going on inside the repeated instructions.

Another problem with a loop is that the values provided for its control cause a different effect than you expected. Discover the difference between what the rules say will happen and what you thought would happen by observing the execution trace, possibly putting in print statements.

Finally, in accumulation loops, the initial value of the accumulator variable may be wrong, or the accumulation is not happening as planned. The control of the loop may stop the accumulation too early, or too late.
### Table 14.16: Troubleshooting loops: a script that doesn't work

<table>
<thead>
<tr>
<th>Troubleshooting loops: a script that doesn't work</th>
</tr>
</thead>
</table>

\[
\begin{align*}
  s & := 0; \\
  x & := .2; \\
  & \text{#Approximate } \exp(x) = 1 + x + x^2/2! + x^3/3! + \ldots \\
  & \text{for } i \text{ to } 10 \text{ from } 1 \text{ by } 1 \text{ do} \\
  & \text{#We changed the order of to and from, but that's not an error} \\
  & \quad s := s + x^i/i!; \\
  & \text{end do;} \\
  & \text{printf("%f compared to expected %f", s, exp(x));}
\end{align*}
\]

\[
\begin{align*}
  0 \\
  0.2 \\
  0.2 \\
  0.2200000000 \\
  0.2213333333 \\
  0.2214000000 \\
  0.2214026667 \\
  0.2214027556 \\
  0.2214027581 \\
  0.2214027582 \\
  0.2214027582 \\
  0.2214027582 \\
  0.2214030000 \\
\end{align*}
\]

0.221403 compared to expected 1.221403

The execution trace seems to indicate that the right number of terms is being computed, but the answer we computed by our own programming is different than what Maple's built-in \( \exp \) function produces. The computed answer seems to be one less than the answer we expect. This is different than the other examples previously shown where the programmed results agreed quite closely with Maple's built-in computation.

### Rerunning the script with more information generated by the execution trace

\[
\begin{align*}
  s & := 0; \\
  x & := .2; \\
  & \text{#Approximate } \exp(x) = 1 + x + x^2/2! + x^3/3! + \ldots \\
  & \text{for } i \text{ to } 10 \text{ from } 1 \text{ by } 1 \text{ do} \\
  & \quad \text{printf("i= %d, x = %f, s = %f, x^i/i! = %f\n", i, x, s, x^i/i!);} \\
  & \quad s := s + x^i/i!; \\
  & \text{end do;} \\
  & \text{printf("%e compared to expected %e", s, exp(x));}
\end{align*}
\]
We enhance the execution trace by having it print out the values of all the variables with each repetition. It's work to enter in this printf statement, but it pays off in helping us find the mistake more quickly. Comparing the actual value of s with what the program comment tells us to expect term by term, we see that while s starts off at zero in the first iteration, the first term \( \frac{x}{i!} \) is .2, not 1 as we expect. Looking at the second and third terms, we realize that we have forgotten to add in the first term to s.

There are two ways to do this -- either to start at i=0 (since \( \frac{x}{0!} \) is 1) or to keep the loop as it is and just initialize s to 1 instead of zero before the loop starts. We opt for the latter since it doesn't require remembering facts from high school days such as that 0!=1.

Table 14.17: Rerunning the script with repairs

| s := 1; |
| x := .2; |
| #Approximate exp(x) = 1 + x + x^2/2! + x^3/3! + ... |
| for i to 10 from 1 by 1 do |
  |  printf("i= %d, x = %f, s = %f, x^i/i! = %f\n", i, x, s, x^i/i!); |
  |  s := s + x^i/i!; |
| end do; |
| printf("%e compared to expected %e", s, exp(x)); |
### 14.9 Summary of Chapter 14

#### Example: Implicit iteration

---

<table>
<thead>
<tr>
<th>Implicit iteration</th>
<th>Example</th>
</tr>
</thead>
</table>
| Using `seq` and `sum` | # Set parameters of script  
|                   | n := 5;  
|                   | # Unassign i just in case it's been used in the session  
|                   | # previously  
|                   | i := 'i';  
|                   | # compute consecutive powers of 10, in a list.  
|                   | tList := [seq(10^i,i=1..n)];  
|                   | # Compute logarithms of the powers you have  
|                   | yList := [seq(evalf(log(tList[i])),i=1..n)];  

Making this change produces the expected agreement. At this point, we would "comment out" the printf that we put in to enhance the execution trace, and could change the value of x to compute an approximation to exp(x) for other values of x.
Do a data plot
plot(tList,yList, style=point, symbol=diamond, color=red,
   labels=["x","log(x)"], symbolsize=30);

Add together the logarithms
s := sum(yList[i],i=1..n):
printf("Sum of logarithms is:\e.\n",s);

Automatic addition
L := [.3, .5];
velocity := .1;
angle := convert(55,units, degrees, radians);
dv := evalf([velocity*cos(angle), velocity*sin(angle)]);
#add dv onto L
newL := L + dv;
### Examples of while loops

**Example of while...do**

```plaintext
#Add on successive terms
while abs(term) >= tol do
  i := i+2;
  term := (-1)^((i-1)/2)*(x^i)/i!;
  s := s+term;
  printf("Term is %e, sum is %f.\n", term, s);
end do: #Suppress execution trace inside the loop #(but printfs still work).

#Print some summary messages.
printf("Sum is: %e compared to %e.\n", s, sin(x));
printf("Stopped when i=%d, term=%e.\n", i, term);
```

```
Sum is: 2.214028e-01 compared to 1.986693e-01.
Stopped when i=11, term=8.888889e-08.
```

**Example of for...to...do**

```plaintext
for i to 10 do
  printf("i= %d, ", i);
end do;
```

```
i= 1, i= 2, i= 3, i= 4, i= 5, i= 6, i= 7, i= 8, i= 9, i= 10,
```

**Example of for...to...by...do**

```plaintext
for i to 10 by 2 do
  printf("i= %d, ", i);
end do;
```

```
i= 1, i= 3, i= 5, i= 7, i= 9,
```

**Example of for...to...from...by...do**

```plaintext
for i to 10 from 5 by 2 do
  printf("i= %d, ", i);
end do;
```

```
i= 5, i= 7, i= 9,
```

### Troubleshooting tips

**Example**

Print out values of variables using execution trace

```plaintext
s := 1;
x := .2;
#Approximate \( e^x = 1 + x + x^2/2! + x^3/3! + \ldots \)
for i to 10 from 1 by 1 do
```

```
```
print("i = %d, x = %f, s = %f, x^i/i! = %f\n", i, x, s, x^i/i!);
  s := s + x^i/i!;
end do;

printf("%f compared to expected %f", s, exp(x));

Have the foresight to avoid running an infinite loop. If despite your best efforts, this happens and hitting the "red stop hand" doesn't work fast enough, kill the Maple session and start again. Hopefully you have saved your work or can use the File->Open Recent->Recover Backup.

# This is "bad news" if you try to run it.
# The condition never becomes false.
# in our loop, so it never stops.
# Some other condition needs to be found
# such as abs(x) >= .001
x := 1.0;
while abs(x)<2 do
  print("sin(x)", sin(x));
  x := x/2;
end do;
15 Conditional Execution with if... end if

15.1 Chapter Overview

We introduce the ability to selectively execute a segment of code through if ... then ... end if. We can select from two alternatives through if...then....else...end if. We can chose code to execute from three or more cases through if...then...elif....elif....else...end if.

Loops and conditional execution are found in most programming languages in a similar form. Learning how to control execution with them is a skill that usually transfers easily, with practice and opportunity, to working with many other computer tools.

15.2 Choosing alternatives: if...then...end if, if... then... else...end if

The if statement allows a programmer to selectively execute code only if a condition is true.

if condition then
code to executed if the condition is true
end if

Table 15.1: If-then-else-end if: one possible outcome

<table>
<thead>
<tr>
<th>If-then-else-end if</th>
<th>one possible outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>grade := 3.7: #This value will cause execution of the &quot;then&quot; code</td>
<td></td>
</tr>
<tr>
<td>if (grade &gt;= 3.5)</td>
<td></td>
</tr>
<tr>
<td>then</td>
<td></td>
</tr>
<tr>
<td>printf(&quot;An excellent grade: %.2f\n&quot;, grade);</td>
<td></td>
</tr>
<tr>
<td>else</td>
<td></td>
</tr>
<tr>
<td>printf(&quot;A Grade: %.2f\n&quot;, grade);</td>
<td></td>
</tr>
<tr>
<td>end if;</td>
<td></td>
</tr>
</tbody>
</table>

An excellent grade: 3.70

If an "else" is added, then the computer will chose which of two segments of code it will execute, depending on whether the condition is true or false.

if condition then
code to executed if the condition is true
else
code to be executed if the condition is false
end if

Table 15.2: If-then-else-end if: another possible outcome

<table>
<thead>
<tr>
<th>If-then-else-end if</th>
<th>another possible outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>grade := 3.2: #This value will cause execution of the &quot;else&quot; code</td>
<td></td>
</tr>
<tr>
<td>if (grade &gt;= 3.5)</td>
<td></td>
</tr>
<tr>
<td>then</td>
<td></td>
</tr>
</tbody>
</table>

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Obviously, if you are planning to do something once, you can probably just look at what's being computed and decide yourself what to do. The value of if is when you plan to use a script many times, or if you need to make many conditional decisions within a loop in a single execution of a script. If is helpful because you won't have to stop to think about what happens each time. You can just set things up ahead of time so that the right actions occur in either case.

Table 15.3: If-then-else-end if in a loop

```plaintext
#Initialize a list with some numbers, some grade point averages
L := [3.6, 3.9, 2.1, 3.9, 2.1, 0.7, 1.3, 2.8, 3.6, 3.7, 2.9];

#Initialize some accumulation variables. We will use them to count how many grades are 3.5 or above, and how many are below.
countHigh := 0;
countLow := 0;

#This goes first, before a loop, as a table header
printf("Grades at least 3.5: \n");

#i steps through the possible index values for L. Recall that nops(L) is the length of the list.
for i from 1 to nops(L) do
    if (L[i] >= 3.5)
        then
            printf("%.2f \n", L[i]); #print out the grade with 2 decimals
            countHigh := countHigh+1; #add one to the "high" counter
        else
            countLow := countLow+1; #add one to the "low" counter
        end if;
    end do:

printf("There were %d grades at or above, and %d grades below 3.5. ", countHigh, countLow);
```
Grades at least 3.5:
3.60
3.90
3.90
3.60
3.70

There were 5 grades at or above, and 6 grades below 3.5.

A fairly common occurrence is when the else alternative would specify to do nothing at all. Instead of having an empty else branch, you should write the code without the else as if...then...endif.

Table 15.4: If-then-end-if

Example of if-then-end-if

In University of West Dakota at Hoople, the Dean's list is determined by computing the average of all student's grade point averages, and selecting those whose gpa is at least .5 point higher than the average.

```plaintext
restart; #unassign any previous values
L := [3.6, 3.9, 2.1, 3.9, 2.1, 0.7, 1.3, 2.8, 3.6, 3.7, 3.8, 2.0]:
deanCount := 0;
#Compute the average for all students by
#using sum to add up all the grades, and dividing by the
#number of grades
avg := sum(L[i],i=1..nops(L))/nops(L):
#Print out table header.
printf("Grade average of %d students is: %f\\n", nops(L), avg);
printf("Grades at least one point above average (Dean's List):\\n");
for i from 1 to nops(L) do
  if (L[i] >= (avg+1) )
    then
      printf("%.2f\\n", L[i]);
      deanCount := deanCount+1;
  end if;
end do:
printf("\nThere were %d students making the Dean's List.\\n",deanCount);
```

Grade average of 12 students is: 2.791667.
Grades at least one point above average (Dean's List):
3.90
3.90
3.80

There were 3 students making the Dean's List.
15.3 Coding situations with three or more cases using if-then-elif-elif...else-end if

The if statements we have seen so far can be described as being useful when:

1. The condition was used to select between one series, and doing nothing as the alternative. (if.. then...endif).

2. The condition was used to select between two alternative series of actions (if...then...else..endif).

Another situation that arises in practice is to select between three or more alternatives. This can be thought of as "case by case" situation.

Table 15.5: if-then-elif-else-endif

<table>
<thead>
<tr>
<th>if-then-elif-else-endif</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>if (condition 1) then</td>
<td>If condition 1 is true when the statement is executed, then the first series of statements is executed.</td>
</tr>
<tr>
<td>statements separated by ; or :</td>
<td>Otherwise, if condition 1 is not true but condition 2 is true, then the second series as executed.</td>
</tr>
<tr>
<td>elif (condition 2) then</td>
<td>Additional conditions will be checked if none of the previous ones turn out to be true. The sequence of actions corresponding to the first true condition will be executed.</td>
</tr>
<tr>
<td>statements separated by ; or :</td>
<td>If none of the conditions are true, then last series of statements are done.</td>
</tr>
<tr>
<td>(more alternatives specified by more elifs)</td>
<td>The else section can be omitted. This is equivalent to doing nothing if none of the other conditions apply.</td>
</tr>
<tr>
<td>else statements separated by ; or :</td>
<td></td>
</tr>
<tr>
<td>end if</td>
<td></td>
</tr>
</tbody>
</table>

The else portion of the statement specifies what is to be done if none of the previous cases apply.

Table 15.6: Example of if-then-elif-elif-...-else-endif if

```
#Initialize L to be a list of grades
L := [3.6, 3.9, 0.9, 2.1, 3.9, 2.1, 0.7, 1.3, 2.8, 3.6, 3.7, 2.9, 1.3];

#Initialize counters foer the number of As, Bs, etc. (silently)
countA := 0;
countB := 0;
countC := 0;
countD := 0;
countF := 0;

#Process each element of the list, with the following cases:
#As (3.3 and above)
#Bs (at or above 2.7 and below an A)
#Cs (at or above 1.7 and
#Ds (at or above 1.0 and below a C)
#Fs (below 1.0).

for i from 1 to nops(L) do
  x := L[i]; #Use "x" instead of typing L[i] several times
  if (x >= 3.3) then
```
countA := countA +1;
elif (x >=2.7) #this is tried only if first condition false
  then
    countB := countB +1;
  endif;
else #we will be here only if everything else hasn't worked
  # so we know x<1.0.
  countF := countF +1;
end if;
end do:
printf("\nThere were %d As, %d Bs, %d Cs, %d Ds, and %d Fs.",
countA,countB,countC, countD,countF);

There were 5 As, 2 Bs, 2 Cs, 2 Ds, and 2 Fs.

It is possible to include if statements as one of the actions inside another if statement. This is useful where a case may have subcases. We will discussion this situation in more detail in later chapters.

15.4 For the curious: comparing control statements in conventional programming languages

Almost all conventional programming languages (C, Mathematica, C#, Java, Matlab, Maple, C++, NXT, Python, Perl, Visual Basic, etc.) have the "loop" idea and the "selection/case" ideas in them. The main difference in how they specify them through the use of different words or symbols. Java for example, uses {} rather than "then" or "end if" to delimit the begin and end of segments of actions.

Table 15.7: Portion of a Java program with a conditional

```java
int countThree, countSubThree;
int [] L= {1,5,9,2,6,7,-2,1,7};
// Set up counters
countThree=0;
countSubThree=0;
// Loop through L counting how many are above three
for (int i=0; i<L.length; i++)
{
  if (L[i]>3)
  { countThree++;}
  else
  {countSubThree++;}
}
System.out.println("three and above:"+countThree+
  " below three: "+ countSubThree);
```
Portion of a Java program with a conditional

While this portion of the Java program is quite intelligible to someone who is more experienced in Maple, a full program to do this would require additional statements that we don't should here. Note that the printing statement \texttt{println} uses "$+" not for numerical addition but to combine the output into a single string that \texttt{println} can handle. Getting used to these language-specific idiosyncrasies are part of the overhead of working with more than one tool on a computer.

The Matlab program follow with a loop and a conditional may look similar to a Maple program, although it uses different functions to extract size ("size" rather than "nops") and there is different syntax (no do, just "end", "+=" instead of \texttt{from} and ".:" instead of to.

Table 15.8: Portion of a Matlab M-file with a loop and a conditional

<table>
<thead>
<tr>
<th>Portion of a Matlab M-file with a loop and a conditional</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\texttt{%Program to count how many elements of a vector/list are greater than and less than three})</td>
</tr>
<tr>
<td>(\texttt{countThree=0;})</td>
</tr>
<tr>
<td>(\texttt{countSubThree=0;})</td>
</tr>
<tr>
<td>(\texttt{L=[1,5,9,2,6,7,-2,1,7];})</td>
</tr>
<tr>
<td>(\texttt{[r,c]=size(L);})</td>
</tr>
<tr>
<td>(\texttt{%Loop for i from 1 to c, where c is the number of items in L (length of L)})</td>
</tr>
<tr>
<td>(\texttt{for i=1:c})</td>
</tr>
<tr>
<td>(\texttt{\hspace{1cm}if L(i)&gt;3})</td>
</tr>
<tr>
<td>(\texttt{\hspace{2cm}countThree=countThree+1;})</td>
</tr>
<tr>
<td>(\texttt{\hspace{1cm}else})</td>
</tr>
<tr>
<td>(\texttt{\hspace{2cm}countSubThree=countSubThree+1;})</td>
</tr>
<tr>
<td>(\texttt{\hspace{1cm}end;})</td>
</tr>
<tr>
<td>(\texttt{end;})</td>
</tr>
<tr>
<td>(\texttt{%Use formatted printing to print out things.})</td>
</tr>
<tr>
<td>(\texttt{\hspace{1cm}sprintf('Three and Above: %d, Below three: %d', [countThree, countSubThree])})</td>
</tr>
</tbody>
</table>

There may be other ways of doing the same task in a language that may be easier in that particular language. For example, summing together numbers in Java involve writing a loop that gathers the sum of the numbers, whereas in Maple one could use "sum" and get the job done in one line. C, Java and C++ have a "?" operator which one could use to write the if of the above example more succinctly even though it would work the same as the kind we have worked about. No language will ever be "the most convenient" for all tasks, which is one of the reasons why so many languages endure.

Regardless of what you think about the way other languages handle loops and conditional execution, learning how to do it in Maple transfers to most other languages you may choose to work with in the future.

15.5 Troubleshooting if statements

While, \texttt{for}, and \texttt{if} statements require you to have words such as "while" "do", "if", "else", etc. written in exactly the right sequence or else the language processor will reject them. Sixty years of advances in programming language technology haven't made the situation for programmers much easier. It's still necessary to learn how to do it correctly, and how to troubleshoot your way out of problems when you make a mistake.

You can use the fact that the computer will let you experiment as much as you want to help you learn. Now that you have learned how to enter code into worksheets and code edit regions, you can start off by learning new syntax by exactly reproducing an example that contains output as well as input, and seeing whether you can get the same results. Next, you can try modifying the examples to do slightly different things, and seeing whether you are successful at the modifications. Finally, you can try to enter something of your own devising, from scratch.

Getting expert help can be a useful way of quickly seeing how to notice and fix the basics. However, the point from getting such help is not to only to get your mistake fixed, but also to learn from the experience so that you can avoid and/or recognize how to fix similar things in the future on your own.
### Table 15.9: Error messages from if statements

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
</table>
| `x := 3;`<br>`countThree := 0;`<br>`countSubThree := 0;`<br>`if x>3`<br>`then`<br>`  countThree := countThree +1;`<br>`else`<br>`  countSubThree := countSubThree +1;`<br>`end if;`<br>`printf("three and above:%d, below three: %d",
  countThree, countSubThree);` | No missing pieces here. |
| `3`<br>`0`<br>`0`<br>`1`<br>`three and above:0, below three: 1` | In this example, we forgot to put in the word "then". After the error, the cursor is positioned in the midst of the line after the "if", which is the first time that Maple's input processor notices that there's something wrong. To handle this error, one looks backwards from the position of the cursor to see if there's anything missing. Admittedly there's not a lot of information given that the cause is a missing `then`, but this is fairly typical in working with programming languages. You'll get an indication that there's something wrong, and the point at which the wrongness was first noticed, but you won't be able to depend on the error message to tell you exact what was wrong. If computer technology could do that reliably, systems would just fix the errors automatically. But doing that well seems to require artificial intelligence beyond present conventional technology. |
| `Error, missing operator or `;` ` |
| `x := 3;`<br>`countThree := 0;`<br>`countSubThree := 0;`<br>`if x>3`<br>`then`<br>`  countThree := countThree +1;`<br>`else`<br>`  countSubThree := countSubThree +1;`<br>`end if;`<br>`printf("three and above:%d, below three: %d",
  countThree, countSubThree);` | We forgot to end the statement with `end if` (or `fi`). |
We said "end do" rather than "end if".

Actually, this is not a mistake. Evidently "end if" and "end" are acceptable here. "fi" is another way of ending an if statement.

Warning, premature end of input, use <Shift> + <Enter> to avoid this message.

Error, reserved word 'do' unexpected

three and above:0, below three: 1
We get the same error message as in the second item in this example. However, this time the problem is that we left off the semi-colon after the end if. We need it if there's an instruction after the if statement in the script.

Programmers need to the consider that there could be more than one kind of mistake that would provoke an error message. This kind of situation can be encountered with almost any sort of programming system (e.g. C, Matlab, Java, etc.) due the limitations of conventional language-processing technology.

```plaintext
x := 3;
countThree := 0;
countSubThree := 0;

if x>3
  then
    countThree := countThree +1;
  else
    countSubThree := countSubThree +1;
end if

printf("three and above:%d, below three: %d",
       countThree, countSubThree);

3
0
0

Error, missing operator or `;`
```

The mistake here is that "IF" does not mean the same thing as "if" to Maple -- case matters. Executing this script gives an error message with the cursor positioned right after the IF. The error message is saying that Maple doesn't know what to do with the IF. It is not particularly informative about the cause. You could discover the cause by comparing what is done with examples that are known to work which all use "if" rather than "IF".

```plaintext
x := 3;
countThree := 0;
countSubThree := 0;

IF x>3
THEN
  countThree := countThree +1;
ELSE
  countSubThree := countSubThree +1;
END IF;

printf("three and above:%d, below three: %d",
       countThree, countSubThree);

3
0
0

Error, missing operator or `;`
```
## 15.6 Chapter summary for Chapter 15

<table>
<thead>
<tr>
<th>If statement summary</th>
<th>Explanation</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>General form</strong></td>
<td>If <em>condition</em> is true when the statement is executed, then the actions are performed, otherwise they aren’t.</td>
<td>if (x &gt;= 3.5) then printf(&quot;%.2f\n&quot;, x); countHigh := countHigh+1; end if;</td>
</tr>
<tr>
<td><strong>Example</strong></td>
<td><em>fi</em> can be used instead of <em>endif</em>. They mean the same thing.</td>
<td></td>
</tr>
<tr>
<td><strong>Explanation</strong></td>
<td>If <em>condition</em> is true when the statement is executed, then the actions are performed, otherwise the alternative actions are performed. <em>fi</em> can be used instead of <em>endif</em>. They mean the same thing.</td>
<td>if (x &gt;= 3.5) then printf(&quot;%.2f\n&quot;, x); countHigh := countHigh+1; else countLow := countLow+1; end if;</td>
</tr>
<tr>
<td><strong>Example</strong></td>
<td><em>elif</em> is a group of statements whose actions are performed if only one condition is true.</td>
<td>if barometer=&quot;falling&quot; and weather=&quot;It's sunny&quot; then weather := &quot;It's cloudy&quot;; elif barometer=&quot;rising&quot; and weather=&quot;It's cloudy&quot; then weather=&quot;It's sunny&quot;; end if;</td>
</tr>
<tr>
<td><strong>Explanation</strong></td>
<td>This is like a &quot;case by case&quot; description of what to do, where the various conditions describe the cases.</td>
<td></td>
</tr>
<tr>
<td><strong>Example</strong></td>
<td>Like the previous situation, only there is an additional set of actions that are done if none of the conditions are true.</td>
<td>#Leaves weather variable unchanged except for two cases where there's change. if barometer=&quot;falling&quot; and weather=&quot;It's sunny&quot; then weather := &quot;It's cloudy&quot;; elif barometer=&quot;rising&quot; and weather=&quot;It's cloudy&quot; then weather=&quot;It's sunny&quot;; end if;</td>
</tr>
<tr>
<td><strong>Explanation</strong></td>
<td>This is like a &quot;case by case&quot; description of what to do, where the various conditions describe the cases, and the last series of actions describe what happens otherwise if none of the other cases hold.</td>
<td></td>
</tr>
<tr>
<td><strong>Example</strong></td>
<td>Like the previous situation, only there is an additional set of actions that are done if none of the conditions are true.</td>
<td>#Determine the location of an (x,y) point. #We assume that x and y already have values from #assignments made in the previous point in execution. if x&gt;=0 and y&gt;=0 then location := &quot;quadrant 1&quot;; elif x&gt;=0 and y&lt;0 then location := &quot;quadrant 4&quot;; elif x&lt;0 and y&gt;=0 then location := &quot;quadrant 2&quot;; elif x&lt;0 and y&lt;0 then location := &quot;quadrant 3&quot;;</td>
</tr>
</tbody>
</table>
end if

else
    location := "unknown";
end if;
15 Conditional Execution with if... end if
16 Thinking about iteration

16.1 Chapter Overview

Looping taps into the power of the computer to do many calculations rapidly. It is not easy to learn but must be mastered in order to be able to handle situations creatively. Knowing standard patterns and roles helps.

Many languages also have shortcuts in common situations that avoid the need to write loops. Using the shortcuts available in Maple (seq, sum, and map, for example) can save programmer time and increase reliability. Other programming languages such as Matlab or Mathematica will have similar shortcuts available.

16.2 Constructing iterations and the roles of variables

Setting up iterations is one of the most prized skills for a programmer, and one of the most difficult for beginners to learn. To construct an iteration from scratch, we recommend the following process:

Without trying to create an iteration, just write down the series of directions for the computational process you want to program. If you keep on writing down steps, eventually you will notice similarities between actions in the series. Often times the similarity allows you to discover that rather than giving a new variable name to every result you compute, you can reuse certain variables because you no longer need them beyond a certain point in the series.

Finding the pattern is key. Once you have it, then the repeating pattern becomes the loop body. Steps before the repetition kicks in become initialization before the loop. Usually there are summary actions (e.g. making a plot or animation, of all the results computed by the loop) which become part of the post-loop actions.

Only through experience do you get to the point where you can construct loops without explicitly finding the repetition pattern is. Don't expect it to come without practice and thoughtful reflection on experience.

In trying to construct a code for a successful loop, it can be good practice to decide upon the role of each variable you are using. If you are unclear about a variable's role or purpose, it may mean that the variable is superfluous. It could also mean that you have skipped over a key computational action which you need to develop code for. Here are typical roles variables can play in a loop:

<table>
<thead>
<tr>
<th>Role Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Variables that are fixed values.</strong></td>
</tr>
<tr>
<td>They are assigned during the initialization but don't change afterwards.</td>
</tr>
<tr>
<td><strong>Variables that are steppers.</strong></td>
</tr>
<tr>
<td>Stepper variables change value methodically with each iteration of the loop.</td>
</tr>
<tr>
<td><strong>Variables that gather the final result.</strong></td>
</tr>
<tr>
<td>Each repetition of the loop leaves the accumulation variable one step closer to its final objective (e.g. adding together all the terms of a sum).</td>
</tr>
<tr>
<td><strong>Variables that hold the most recent value going through a succession.</strong></td>
</tr>
<tr>
<td>Usually (but not always) the most recent value is computed from previous values.</td>
</tr>
<tr>
<td><strong>Variables that hold the previous value.</strong></td>
</tr>
<tr>
<td>Sometimes in order to compute the next value, it is necessary to keep not only the present value, but the one before that. Since a variable can hold only one value at a time, an extra variable needs to be used to play that role. The examples given so far do not have any variables with this role, but we will soon see situations where this is needed.</td>
</tr>
</tbody>
</table>

Consider for example this starter code from Lab 2, that computed chemical concentrations

<table>
<thead>
<tr>
<th>Table 16.2: Lab 2 CS 122 Problem 2.1 starter code</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lab 2 CS 122 Chemical problem starter code</td>
</tr>
<tr>
<td>restart;  #just in case there are unwanted assignments hanging around from #previous runs.</td>
</tr>
<tr>
<td># Chemical reaction modelled by Lotka-Volterra Model #Discrete approximation</td>
</tr>
</tbody>
</table>
numTimeSteps := 50;

#Initial concentrations
A0 := 1000;
B0 := 0;
X0 := 10;
Y0 := 50;
#Chemical constants
k1 := .001;
k2 := .001;
k3 := .1;
Atab := table();
indexTab := table();
A := A0; B := B0; X := X0; Y := Y0;

for i from 1 to numTimeSteps do
  #Calculate next values of chemical concentrations
  newA := A - k1*A*X;
  #newX :=
  #newY :=
  #newB :=
  #Update all four chemicals
  A := newA; #B, X, Y updated similarly.
  indexTab[i] := i;
  Atab[i] := A; #Each entry is a (time, concentration)
end do:

with(plots): with(plottools):
#Given a table of x coordinates and a table of y coordinates,
#Create a point plot.
Fplot := (itab,vtab,COLOR) ->
plot(convert(itab,list),convert(vtab,list), color=COLOR, symbol=diamond, style=point);
#Display this point plot, along with a title and labels.
display([Fplot(indexTab,Atab,"Green")], title="graph of A", labels=["time", "concentration"]);

The analysis of the roles of variables in this code is as follows:

Table 16.3: Roles of variables in a loop

| Variables that are fixed values | k1, k2, k3 are fixed values. So are A0, B0, X0, and Y0. So is numTimeSteps. |
| Variables that are steppers | i is a stepper variable. It begins at 1. One is added onto its value each trip through the loop, so it steps through the values 1, 2, 3, .... |
| Variables that gather the final result | While there are no variables that sum or count in this example, indexTab and Atab are tables that store one value per iteration. The tables gather the whole collection of values generated during the simulation, which are needed to produce the plot which is the final result. |
| Variables that hold the most recent value going through a succession | newA is holds the most recent value of the concentration of chemical A. newB, newX, and newY also take on this kind of role. |
| Variables that hold the previous value | At the beginning of the loop, A (and as the code was developed, B, X, and Y) took on the role of the previous value. It was used to compute |
the one before that. Since a variable can hold only one value at a time, an extra variable needs to be used to play that role. The examples given so far do not have any variables with this role, but we will soon see situations where this is needed.

newA, newB, etc. At the end of the loop body, A is set to the most recent value in preparation for the next iteration of the loop when those values will once again become "previous".

Here's a classic example of a task accomplished through a loop: finding the smallest value of a list of numbers

**Table 16.4: An example using a variable that holds the previous value**

<table>
<thead>
<tr>
<th>An example using a variable that holds the previous value</th>
</tr>
</thead>
<tbody>
<tr>
<td>#We want to find the smallest value in a list of numbers</td>
</tr>
<tr>
<td>L := [3, 6, -4, 2, 1, 9];</td>
</tr>
<tr>
<td>#minSoFar's role is an accumulator variable:</td>
</tr>
<tr>
<td># it keeps track of the &quot;smallest number so far&quot;. We initialize it to infinity</td>
</tr>
<tr>
<td># so that whatever we see first will automatically become the smallest so far.</td>
</tr>
<tr>
<td>minSoFar := infinity;</td>
</tr>
<tr>
<td>#Step through all the positions of L: 1, 2, 3... 6</td>
</tr>
<tr>
<td>for i from 1 to nops(L) do</td>
</tr>
<tr>
<td>printf(&quot;Comparing %a to %d\n&quot;, minSoFar, L[i]);</td>
</tr>
<tr>
<td>#If the previously seen minimum is larger than L[i]....</td>
</tr>
<tr>
<td>if minSoFar&gt;L[i]</td>
</tr>
<tr>
<td>#Make it the new previously computed minimum</td>
</tr>
<tr>
<td>then minSoFar := L[i];</td>
</tr>
<tr>
<td>end if;</td>
</tr>
<tr>
<td>end do;</td>
</tr>
<tr>
<td>printf(&quot;The minimum is: %a.&quot;, minSoFar);</td>
</tr>
</tbody>
</table>

```
[3, 6, -4, 2, 1, 9]

Comparing infinity to 3
Comparing 3 to 6
Comparing 3 to -4
Comparing -4 to 2
Comparing -4 to 1
Comparing -4 to 9
The minimum is: -4.
```

The analysis of the roles of variables in this code is as follows:

**Table 16.5: Roles of variables in a loop**

<table>
<thead>
<tr>
<th>Variables that are fixed values. They are assigned during the initialization but don't change afterwards.</th>
</tr>
</thead>
<tbody>
<tr>
<td>L is the only fixed variable.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variables that are steppers. Stepper variables change value methodically with each iteration of the loop.</th>
</tr>
</thead>
<tbody>
<tr>
<td>i is a stepper variable. It begins at 1. One is added onto its value each trip through the loop, so it steps through the values 1, 2, 3, ....</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variables that gather the final result.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Each repetition of the loop leaves the accumulation variable one step closer to its final objective (e.g. adding together all the terms of a sum).</td>
</tr>
<tr>
<td>There aren't any &quot;gatherers&quot; in this short bit of code.</td>
</tr>
</tbody>
</table>
### Variables that hold the most recent value

Going through a succession, usually (but not always) the most recent value is computed from previous values.

---

### Variables that hold the previous value

Sometimes in order to compute the next value, it is necessary to keep not only the present value, but the one before that. Since a variable can hold only one value at a time, an extra variable needs to be used to play that role. The examples given so far do not have any variables with this role, but we will soon see situations where this is needed.

---

### 16.3 Reminder: you can skip writing some looping code through use of built-in operations

As we mentioned in chapter 14 (page 209), Maple has built-in operations such as `sum` or `seq` that can help users avoid the work of writing looping code because they have looping built into them. For example, `min([3, 6, -4, 2, 1, 9])` will compute the minimum without having to write any loops at all. Thus, Maple users would probably not find minimums using the "classic example" of the previous section. Using `min` would be faster.

Similarly Maple users would use `sum(L[i], i=1..nops(L))` to add together the elements of L without a loop.

#### Table 16.6: Generating a multi-plot with `seq`

<table>
<thead>
<tr>
<th>Plot $\frac{\sin(i \cdot x)}{i}$ for $i = 1..4$ altogether on the same plot.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i := 'i'$</td>
</tr>
<tr>
<td>$i := i$</td>
</tr>
<tr>
<td>$\text{Lcolors} := \text{&quot;Red&quot;, &quot;DarkGreen&quot;, &quot;Blue&quot;, &quot;Magenta&quot;}$</td>
</tr>
<tr>
<td>$\text{Lcolors} := \text{[&quot;Red&quot;, &quot;DarkGreen&quot;, &quot;Blue&quot;, &quot;Magenta&quot;]}$</td>
</tr>
<tr>
<td>$\text{Lplots} := \text{seq} \left( \text{plot} \left( \frac{1}{i} \cdot \sin(i \cdot x), x = 0 \ldots 10, \text{color} = \text{Lcolors}[i] \right), i = 1..4 \right)$</td>
</tr>
<tr>
<td>$\text{Lplots} := { \text{PLOT}(...), \text{PLOT}(...), \text{PLOT}(...), \text{PLOT}(...) }$</td>
</tr>
</tbody>
</table>

We unassign `i` so that it will work with the `seq` that we're about to do. Alternatively, we could use the instruction `restart` if we didn't mind losing all the assignments we've already made in the Maple session.

We create a list of colors taken from the official list of Maple colors (see on-line help for `colornames`).
Here is another programming feature which you can use to cut down on the amount of work that you would normally do with a loop. A common situation in programming is when you want to perform the same operation on each component of a structure. We could do this with a loop or with seq, but there's an even shorter way to do it in Maple, \textit{map}.

\texttt{map(function or function name, structure)}

will apply the function to each element of the structure, producing a new structure where the \(i\)-th element of the new structure is the result of applying the function to the \(i\)-th element of the old structure.

The advantage of \textit{map} over loops or \texttt{seq} is that the programmer does not need to set up any stepper variables, nor need to figure out how to construct the same kind of structure that the input has. The programmer does not need to use \texttt{nops} or other techniques to figure out the length of the structure, as \textit{map} will figure out the length automatically.

The typical use of \textit{map} is to a list or set. \textit{map} does not work with sequences because the use of a sequence does not present \textit{map} with a single structure to map onto.

\textit{map} does not work on tables. It will however, work, if you convert a table into a list and then apply \textit{map} to the list.

Table 16.7: Generating a multi-plot with seq

\begin{verbatim}
L := [1.2, 3, \sin\left(\frac{\pi}{3}\right)]
\end{verbatim}

\begin{verbatim}
[1.2, 3, \frac{1}{2} \sqrt{3}]
\end{verbatim}

We find the absolute value of every number in L by having \textit{map} apply \texttt{abs} to every element of L. We create a new list of the absolute values and assign it to L2.

Table 16.7: Generating a multi-plot with seq

| \texttt{L := [1.2, 3, \sin\left(\frac{\pi}{3}\right)]} | \texttt{\[1.2, 3, \frac{1}{2} \sqrt{3}\]} | We find the absolute value of every number in L by having \textit{map} apply \texttt{abs} to every element of L. We create a new list of the absolute values and assign it to L2. |
We find the natural log of every number in set1. Since they are all floating point numbers, applying ln to them results in approximations to the logarithms.

We get a list of left hand sides of the equations in list3.

Mapping plot onto these expressions produces a list of plot structures. We can plot them together using display.

If we’re feeling confident, we can daisy-chain (compose) the functions together into a one line of coding that takes list6 and produces a multi-plot from it. You’d probably develop the code for this working inside out. First you’d convince yourself that the lhs map worked. Then you’d add on the plot. Finally, you’d add on the display and the parameters that control the range of horizontal and vertical views. It would take multiple edits, but you would only have to execute a single line of code repeatedly, rather than executing an entire worksheet or selecting a segment to re-execute.
A variety of programming languages (e.g. Lisp, Matlab, Python, Mathematica) have map or a similar technique of computation. For example, Matlab has ARRAYFUN(f, A) that applies a function f to every element of a Matlab array (a kind of data container) A. map is a key component of Google's "easy supercomputing" package (see http://code.google.com/edu/parallel/mapreduce-tutorial.html).

16.5 The advantages and limitations of shortcuts

Shortcuts are good because:

1. They save programmer time, once the programmer masters the shortcut.

2. In Maple they typically are more efficient because more of the computation is handled by internal programming which typically has been optimized for speed.

3. One theory of software engineering is that long programs can be harder to debug, and have more bugs in them. To the extent that shortcuts reduce the length of programs, they can save programmer time because there's less time spent debugging.

Shortcuts such as seq or map have limitations, however. Not all computational tasks are shortened by using such techniques, particularly when you have many variables with "previous value" roles and don't have enough memory or time to construct lists or tables that hold all the previously computed results. There will still be many situations where a for/while loop is the best way of doing the computation.
16.6 Summary of chapter

<table>
<thead>
<tr>
<th>Variable types</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed value</td>
<td>These variables are set before the loop begins, and do not change while the loop is being executed.</td>
</tr>
<tr>
<td>Steppers</td>
<td>These variables change some fixed amount every time a loop iterates. They hold the current step a loop is on. For example, the index variable in a for loop is a stepper variable.</td>
</tr>
<tr>
<td>Variables that gather the final result</td>
<td>These variables accumulate a result that is returned at the end of a loop.</td>
</tr>
<tr>
<td>Variables that hold the most recent value in a loop</td>
<td>These variables hold some value that was computed during the current iteration of a loop. They are often compared to a previous value to compute a change.</td>
</tr>
<tr>
<td>Variables that hold a previous value in a loop</td>
<td>These variables hold some value computed in the previous iteration of a loop.</td>
</tr>
</tbody>
</table>

Looping without using built-in operations

Plot \( \frac{\sin(i \cdot x)}{i} \) for \( i = 1..4 \) altogether on the same plot.

\[
i := 'i'
\]

\[
Lcolors := \text{"Red", "Blue", "Pink", "Black"}
\]

\[
\text{"Red", "Blue", "Pink", "Black"}
\]

\[
Lplots := \text{seq} \left( \text{plot} \left( \frac{1}{i} \cdot \sin(i \cdot x), x = 0..10, \text{color} = Lcolors[i], i = 1..4 \right) \right)
\]

\[
\text{[PLOT(...), PLOT(...), PLOT(...), PLOT(...)]}
\]

\[
\text{display}(Lplots)
\]

Using the map function


Performing an operation on every item in a set.

\[ set1 := \{1, 0.0, 2.2, 9.5\} \]

\[
(1.0, 2.2, 9.5) \quad (16.16)
\]

\[ map(ln, set1) \]

\[
(0.0, 0.7884573604, 2.251291799) \quad (16.17)
\]

Solving a list of equations.

\[ list3 := \left[ x^2 - 1 = 0, x^3 - 1 = 0 \right] \]

\[
[x^2 - 1 = 0, x^3 - 1 = 0] \quad (16.18)
\]

\[ map(solve, list3) \]

\[
[1, -1, 1, -\frac{1}{2} + \frac{1}{2} 1\sqrt{3}, -\frac{1}{2} - \frac{1}{2} 1\sqrt{3}] \quad (16.19)
\]

Solving a set of equations. Note that duplicates are removed.

\[ set3 := \text{convert}(list3, \text{set}) \]

\[
\{x^2 - 1 = 0, x^3 - 1 = 0\} \quad (16.20)
\]

\[ map(solve, set3) \]

\[
[-1, 1, -\frac{1}{2} - \frac{1}{2} 1\sqrt{3}, -\frac{1}{2} + \frac{1}{2} 1\sqrt{3}] \quad (16.21)
\]

Using map to display a set of plots.

\[ list4 := map(lhs, list3) \]

\[
[x^2 - 1, x^3 - 1] \quad (16.22)
\]

\[ list5 := map(plot, list4) \]

\[
[PLOT(...), PLOT(...)] \quad (16.23)
\]

\[ with(plots) : \]
display(list5)
17 Maple procedures: writing programs as functions

17.1 Chapter Overview

We have already learned how to create user-defined functions through use of -> in Chapter 7.

In this chapter, we introduce an alternative way of defining a function that allows the use of programming statements, e.g. if, for, or while through the use of

\[
\text{function name} := \text{proc}(...)
\]

\[
\text{statement;}
\]

\[
\ldots
\]

\[
\text{statement;}
\]

\[
\text{end proc;}
\]

This allows us to develop more complicated functions that use such statements. We can encapsulate the definition and use of variables within the procedure (local variables) by the alternative form

\[
\text{function name} := \text{proc}(...)
\]

\[
\text{local} \text{ variable names;}
\]

\[
\text{statement;}
\]

\[
\ldots
\]

\[
\text{statement;}
\]

\[
\text{end proc;}
\]

17.2 Review of function terminology

Let's review what we know function terminology in Maple, as has been developed in the chapters so far.

Table 17.1: Review of function terminology

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
</table>
| 1. Functions process input values and produce an output value. The function definition describes how the values of those parameters are used to compute a function result. When we defined functions using the -> (arrow) notation, the result was described as an expression involving arithmetic, structure creators such as \[ \], and built-in (e.g. \text{sin}, \text{sqrt}, or \text{seq}) functions or functions previously defined by the user. | \[
f := (a, b) \rightarrow \sqrt{a+b}
\]

\[
(a, b) \rightarrow \sqrt{a+b}
\]

There are two formal parameters to the function defined above. \(a\), and \(b\) are their name. |
| 2. Functions are associated by a name by making the function definition the right hand side of an assignment statement. | The function \((a, b) \rightarrow \sqrt{a+b}\) is named \(f\) through the assignment. |
| 3. Defining a function does not perform the computation, it just stores the directions for how to do the computation within the computer. After a function is defined, you can get the computer to perform the computation by invoking the function. Invocation involves giving the name of the function, and values for the function's arguments, enclosed in parentheses. The values supplied to the function when it is invoked are called the actual parameters or arguments to the function. | \[
f(3.1, \pi) \ # \text{The actual parameters (arguments)}
\]

\[
# \text{are 3.1 and } \pi.
\]

\[
1.760681686 \sqrt{\pi}
\]

(17.2) |
4. We can combine the actions of several functions together by composing (also known as daisy-chaining) them. Rather than the "one after another" sequencing of actions that we might get from using functions on several lines of a script, function composition uses "inside out" sequencing.

\[
\text{max}(f(3.1, 6), f(4.7, 3)) = 4.312771731
\]

(17.3)

This is the same as the sequence

\[
\begin{align*}
\text{result1} & := f(3.1, 6); \\
\text{result2} & := f(4.7, 3); \\
\text{result3} & := \text{max}(\text{result1}, \text{result2})
\end{align*}
\]

\[
\begin{align*}
4.312771731 \\
3.754996671 \\
4.312771731
\end{align*}
\]

(17.4)

Programming languages feature functions prominently because of their advantages to the construction of programs. Writing a computation as a function encapsulates it: there is a well-defined starting point given by the values supplied for the parameters, there is a specific computation described within the function definition, and there is a final result to the computation when the function delivers its output. Only when the function is later invoked are those actions performed. The function may be invoked multiple times (as would occur when the users invokes it when doing several problems, or when map is used with a function) after it is defined. The expectation of reuse is one of the justifications for the extra work needed to define a function, since re-invocation is typically easier than even copying and pasting the actions.

### 17.3 Software engineering advantages of functions versus function-less scripts

Almost all programming languages use functions (sometimes called procedures, and, in the case of object-oriented languages such as C++ or Java, called methods.) Why is this feature so popular?

1. Daisy-chaining functions allows one to build complex behavior a piece at a time. Each piece can be separately tested before linked together with the next piece. This makes testing and debugging simpler because checking several small pieces individually for an error is a much faster process than looking at a single large piece. Simpler development means lower costs for programming more complex tasks.

2. Functions encapsulate an action. There is a clear beginning and end to the statements that provide the action, making it easier to transfer to other scripts or worksheets. The input to the function is explicitly named at the start -- the formal parameters to the function. This makes the function easier to reuse than an ordinary script. Reuse lowers the cost of programming because it allows you to reuse the same programming on multiple jobs. Ease of reuse means lower costs for programming more complex tasks.

3. Functions give a name to an action, and allow you to invoke the action just by giving its name and the arguments that it should be applied to. This reduces the amount of work, compared to repeated copying, pasting, and editing if the function is used multiple times. Again, this makes the programming written easier to reuse and therefore brings lower costs for programming more complex tasks.
17.4 Creating a short function with proc... end

One line functions created by -> have the limitation that we can't create functions that have code which includes assignment, loops, or conditionals in them. We now discuss another way of creating functions in Maple that allow these things.

Table 17.2: General form of a Maple procedure definition

| function name := proc (formal parameters) |
| statements; |
| return expression or variable containing result of the function; |
| end proc; |

Notes
1. The statements can include if, and for but not any that use assignment.

Table 17.3: A simple function created with -> and proc

\[
\begin{align*}
    f &:= (a, b) \rightarrow \frac{\sqrt{a-b}}{2} \\
    (a, b) &\rightarrow \frac{1}{2} \sqrt{ab} \\
    f(3.0, \pi) &\rightarrow 0.8660254040 \sqrt{\pi}
\end{align*}
\]

17.5 Creating a function with an if statement in it

An important feature of the proc...end proc way of creating functions is that the code inside the function definition can contain if, while, and for statements.

Table 17.4: A short function created by proc end with a control statement

\[
\begin{align*}
    f &:= \text{proc}(t, a, b, c); \\
    \text{if } t > a \text{ then return } b; \\
    \text{else return } c \cdot t; \\
    \text{end if}; \\
    \text{end proc;} \\
    \text{proc}(t, a, b, c) \rightarrow \frac{1}{2} \sqrt{a \cdot b} \text{ end proc}
\end{align*}
\]

We want to model a situation where a sprinter in a race accelerates at a rate of five meters per second squared for the first two seconds of a race, then travels at a constant velocity of 10 meters/second. \( t \) is the time, \( a \) is the cutoff point when acceleration stops, \( b \) and \( c \) are the parameters that help to describe the velocity before and after time \( a \).
This shows the velocity of the runner at 1 second, 3 seconds, and 1.3 seconds.

\[(f(1, 2, 10, 5) \quad \begin{array}{c} 5 \\ \text{(17.10)} \end{array} \]

\[(f(3, 2, 10, 5) \quad \begin{array}{c} 10 \\ \text{(17.11)} \end{array} \]

\[(f(1.3, 2, 10, 5) \quad \begin{array}{c} 6.5 \\ \text{(17.12)} \end{array} \]

The advantage of designing \( f \) to use the parameters \( a, b, c \) in designing the function is it makes us to model similar sprinters without changing the programming of \( f \). For example, if we had a sprinter who accelerates at a rate of three meters per second squared for the first three seconds, then travels at a constant velocity of nine meters per second, we can use \( f \) to determine the velocity of the sprinter at \( t = 1.4 \) seconds in the way illustrated.

This attempt to use the arrow notation to define a function with an "if" fails miserably. You can see that one of the problems is that the computer will get confused because the end of the function definition does not end with the first semi-colon.

```
f2 := (t, a, b, c) \rightarrow \text{if } t > a \text{ then return } b; 
\text{else return } c \cdot t; 
\text{end if};
```

Error, invalid arrow procedure

```
f2 := (t, a, b, c) \rightarrow \text{if } t > a \text{ then return } b; 
\text{else return } c \cdot t; 
\text{end if};
```

### 17.6 Creating a function with local variables

Table 17.5: General form of a Maple procedure definition

<table>
<thead>
<tr>
<th>function name := proc (formal parameters)</th>
</tr>
</thead>
<tbody>
<tr>
<td>local variables used within procedure;</td>
</tr>
<tr>
<td>statements;</td>
</tr>
<tr>
<td>return expression or variable containing result of the function;</td>
</tr>
<tr>
<td>end proc;</td>
</tr>
</tbody>
</table>

**Notes**

1. There is no semi-colon between the proc ( ) containing the formal parameters and the word local. If there are no local variables for the procedure, then you need to put in the semi-colon: proc (formal parameters); statements; ....

2. The opposite of a "local variable" is a "global variable". We don't use them much in the Maple procedures we will be working with. You can read more about them if you look up "global" in Maple on-line help.

3. You can't assign a value to a formal parameter in the statements of the procedure. parameters are used only as a way of providing input information to the function.

Usually local variables arise when the function needs to compute and store intermediate results that are used for the final answer but are not part of them.
A function with local variables

Table 17.6: A function with local variables

<table>
<thead>
<tr>
<th>A function with local variables</th>
<th>These are lists whose every item is a list of a state name and a capital city name.</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>LStates1 := [ [&quot;Pennsylvania&quot;, &quot;Harrisburg&quot;], [&quot;Massachusetts&quot;, &quot;Boston&quot;], [&quot;California&quot;, &quot;Sacramento&quot;], [&quot;Florida&quot;, &quot;Tallahassee&quot;], [&quot;Texas&quot;, &quot;Austin&quot;]];</code></td>
<td>The utility of this function is that we can reuse this kind of max-finding very easily for any number of lists of states and capitals. The last line shows how we could get the result to be printed out without the quotation marks.</td>
</tr>
</tbody>
</table>

```
#This function when given a list of pairs of strings, #compares the length of the second item of each pair and #finds the longest. It then prints out the corresponding #first item.
findLongest := proc (LL)
    local i, maxIndex, maxSoFar;
    maxIndex := 0;
    maxSoFar := 0;
    for i to nops(LL) do
        if maxSoFar < length(LL[i][2])
            maxIndex := i; #position
            maxSoFar := length(LL[i][2]); #number of letters
        end if;
    end do;
    return cat("State with longest capital name is: ", LL[maxIndex][1]);
end proc;
```

"raw" result of invoking procedure

```
findLongest(LStates1);
```

#result of daisy chaining procedure with printf to # pretty-print result

```
printf(%s,findLongest(LStates2));
```

```
proc(LL)
    local i, maxIndex, maxSoFar;
    maxIndex := 0;
    maxSoFar := 0;
    for i to nops(LL) do
        if maxSoFar < length(LL[i][2]) then
            maxIndex := i; maxSoFar := length(LL[i][2])
        end if;
    end do;
    return cat("State with longest capital name is: ", LL[maxIndex][1])
end proc
```

"State with longest capital name is: Florida"

State with longest capital name is: Saarland

An important feature of local variables is that use of their use inside the function is encapsulated. When a variable is declared to be a local variable of a function, any values that the symbol may have is temporarily set aside. The symbol within the function starts off with no value, but may be assigned values within the programming of the function. Once the function invocation is over, the "old" value that the symbol had is restored.
This allows the programmer to use a symbol inside a function without having to worry about what that symbol is being used for before or after the function is invoked. Without this feature, it would be difficult to reuse functions because users would be forbidden from including in their scripts any symbol that one of the functions used, because it would break the programming of either the script or the function.

Table 17.7: Demonstration that local variables have no connection with "outer"/global symbols of the same name

<table>
<thead>
<tr>
<th>Demonstration that local variables have no connection with &quot;outer&quot;/global symbols of the same name</th>
<th>We define findLongest2, whose only difference from the previously defined function findLongest is that it prints out the value of the local variable i.</th>
</tr>
</thead>
</table>

```maple
#This function when given a list of pairs of strings, compares
#the length of the second item of each pair and finds the longest.
#It then prints out the corresponding first item.
findLongest2 := proc (LL)
    local i, maxIndex, maxSoFar;
    maxIndex := 0;
    maxSoFar := 0;
    for i to nops(LL) do
        if maxSoFar < length(LL[i][2]) then
            maxIndex := i; #position
            maxSoFar := length(LL[i][2]); #number of letters
        end if;
    end do;
    printf("i within findLongest after the loop is:%a\n",i);
    return cat("State with longest capital name is: ", LL[maxIndex][1]);
end proc:
```

```maple
i := 'i';
printf("i before invocation of findLongest2 is:%a\n",i);
i before invocation of findLongest2 is: i

findLongest2(LStates1);
i within findLongest after the loop is: 6
"State with longest capital name is: Florida" (17.16)

printf("i after invocation of findLongest2 is:%a\n",i);
i after invocation of findLongest2 is: i

i := 47;
i after invocation of findLongest2 is: i
```

We first make sure that i is unassigned -- has no value.

We are in a situation where i is being used in two different ways -- in the script, and as a local variable in the function.

The first line of output by the script shows that the script's version of i is just a symbol, since it has not yet been defined.

After returning from the function invocation, we print out the value of i. Since this is the script version of i, its value is as it was before we invoked findLongest2 -- unassigned.

We now assign the script version of i.
17.7 Designing troubleshooting into your functions

Maple, like many languages has a statement that allows the programmer to stop operation of the program if a situation that shouldn't arise actually happens. The general form of the statement is

\[ \text{error, sequence of expressions;} \]

When this statement is executed, Maple generates an error message. By default, this causes a message of the form Error, ... is printed out and execution halts immediately. The message is the value of the execution will result of the sequence of expressions (usually some strings describing the nature of the situation, or the values of variables where something funny is going on.) Typically the error statement occurs in an if statement designed to test for a potentially erroneous or unintended situation arising.

Building in if statements in a function checking that all is well, but printing out an error message if it is not, is a good way of reassuring yourself that your coding is working as you intended. Because execution stops as soon as the error message is printed, it can reduce situations where you have to wade through many lines of an execution trace trying to find where an error occurred.

Table 17.8: A Maple function created with error signaling

<table>
<thead>
<tr>
<th>A Maple function created with error signaling</th>
<th>This short script demonstrates what we want to happen -- if the state of the fan is high (the symbol), then we want to assign the fanSpeed to 5000. If the state of the fan is low, then, we want fanSpeed to be 2000.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{fanState := high,} )</td>
<td>( \text{speedFast := 5000;} ) \hspace{1cm} \text{(17.19)} ) \text{high} \hspace{1cm} \text{(17.20)} \text{5000} \hspace{1cm} \text{speedSlow := 2000;} ) \hspace{1cm} \text{(17.21)} \text{2000} \hspace{1cm} \text{if \text{fanState = high}} ) \hspace{1cm} \text{then fanSpeed := speedFast,} ) \hspace{1cm} \text{end if,} \hspace{1cm} \text{if \text{fanState = low}} ) \hspace{1cm} \text{then fanSpeed := speedSlow,} ) \hspace{1cm} \text{(17.22)} \text{5000} | The way this code is written, if fanState were not set to high nor low, then the fanSpeed is not set. If subsequent steps of the script used the value of fanSpeed and assumed that it had been assigned by this, we would start seeing mysterious problems arising.</td>
</tr>
<tr>
<td>#Returns the value of a fast or slow speed for a fan #based on the fan state, which should be either the #symbol high or low.</td>
<td>#setSpeed := proc (fanState) #local speedFast, speedSlow; #speedFast := 5000; #speedSlow := 2000;</td>
</tr>
<tr>
<td>We have designed a function that gives an error message that is intelligible by design.</td>
<td>When we build this into a function, we add a little error-detection into the function itself, so that it will abort operation of whatever code it is running in if it ever encounters a fanState that is neither high nor low.</td>
</tr>
</tbody>
</table>
Putting a colon after the last procedure suppresses printing the function when the code region is executed.

```maple
if fanState = high
then return speedFast
elif fanState = low
then return speedSlow
else error "fanState is", fanState, "should be high or low"
end if
end proc;

setSpeed(high);
setSpeed(low);
setSpeed(middle);

proc(fanState)
    local speedFast, speedSlow,
    speedFast := 5000;
    speedSlow := 2000;
    if fanState = high then
        return speedFast
    elif fanState = low then
        return speedSlow
    else
        error "fanState is", fanState, "should be high or low"
    end if
end proc

5000
2000

Error, (in setSpeed) fanState is, middle, should be high or low
```

### 17.8 Troubleshooting small function definitions

Function definitions with proc... end proc will exhibit typical "imbalance" error messages such as errors if you forget to balance the typical things that occur in computer syntax: ( and ), if and end if, [ and ], proc and end proc. If you do it in a code edit region, then in addition to printing out an error message, the cursor will pause over the first place where the computer saw the imbalance.

**Table 17.9: A Maple function created with error signaling**

<table>
<thead>
<tr>
<th>Troubleshooting problems with small function definitions</th>
<th>Seeing this message is an indication that you didn't balance what you started.</th>
</tr>
</thead>
<tbody>
<tr>
<td>f := proc (t, a, b, c) if a &gt; t then return b else return c*t ;</td>
<td>We note that there's no &quot;end proc&quot; to end the procedure definition.</td>
</tr>
<tr>
<td>Warning, premature end of input, use &lt;Shift&gt; + &lt;Enter&gt; to avoid this message.</td>
<td>After making that correction, we get a message that indicates that the proc wasn't expected. The cursor is positioned in the midst of the &quot;end proc&quot; so we know that it's the last proc where things ran into problems.</td>
</tr>
</tbody>
</table>
We know that procedures must end with `end proc`, so removing it doesn't seem to be an option. But another cause of `end proc` being unexpected is that something else was expected instead.

Looking harder, we realize, that we forgot the "end if" to end the if statement, too.

We make that correction, and have no error or warning messages. But we discover that the results are not the same as the previous section.

This is not a balancing problem, but an indication that we are giving incorrect instructions. Looking at the evidence and at the code carefully, we discover that we didn't get the inequality correct.

This seems to get it right. It's good to have enough test to thoroughly check out the tests.
Troubleshooting problems with use of variables in procedures

This is a function that's supposed to take \( n \) square roots of a number:

\[
\sqrt[n]{\sqrt[n]{\sqrt[n]{a}}} \quad \text{There are several problems with it. The first problem is that in order to declare a local variable, Maple insists that there be no extra semi-colon between the parameters (a,n) and the word "local". It should be proc(a,n) local i; ...}
\]

With this corrected, we get Maple to accept the definition, but it still doesn't work. The problem is that formal parameters can't be assigned to in Maple procedures. Formal parameters do not work the same as local variables.

The way to get the desired result is to set up a local variable and to initialize it with the value of a before the loop starts.

The function works both for exact numbers and floating point numbers.
17.9 Creating a function from a script

In a typical script, we start with assigning certain variables values. We have called these variables *formal parameters* in that once assigned, we use them through the rest of the script. We can turn a script into a Maple procedure by a process that turns the script parameters into the parameters of the procedure, and the other variables used in the script into the *local variables* of the procedure. We may have to modify the script so that we avoid breaking the rule that we can't assign symbols that are used as formal parameters for the procedure. We illustrate this process through the following example.

**Table 17.10: A script to compute a sum**

<table>
<thead>
<tr>
<th>A script to compute a sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>The original, from chapter 12, computed a sum that approximated $e^x$ and printed out a comparison of the sum with Maple's value for $e^x$. This modification just calculates the final value.</td>
</tr>
</tbody>
</table>

```plaintext
# Compute a sum $x^i/i!$ until the term < tol.

tol := 10e-10;
x := .05;
s := 1; # accumulates the sum
term := 1;
for i from 1 while abs(term)>=tol do
  term := term*x/i;
s := s+term;
end do:
s;
```

We identify the parameters as $tol$, and $x$. The local variables are $s$, $i$ and $term$. The result is in $s$ after the loop is finished. This leads us to transform the script into a function in this way:

**Table 17.11: A procedure to compute a sum**

<table>
<thead>
<tr>
<th>A procedure to compute a sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>eApprox := proc(x, tol)</td>
</tr>
<tr>
<td>local s, i, term;</td>
</tr>
<tr>
<td>s := 1;</td>
</tr>
<tr>
<td>term := 1;</td>
</tr>
<tr>
<td>for i from 1 while abs(term)&gt;tol do</td>
</tr>
<tr>
<td>term := term*x/i;</td>
</tr>
<tr>
<td>s := s+term;</td>
</tr>
<tr>
<td>end do;</td>
</tr>
<tr>
<td>return s;</td>
</tr>
<tr>
<td>end proc;</td>
</tr>
</tbody>
</table>
The assignment names the function eApprox. We can invoke the function in a script that compares the results of the function with Maple's computed value. By separating the definition of the function from the definition of the printing loop, we make both easier to comprehend. We can tell at a glance that the printing loop prints out the result of eApprox.

**Table 17.12: Invoking the procedure after it’s been defined**

<table>
<thead>
<tr>
<th>i</th>
<th>eApprox(i, 10^{-5})</th>
<th>exp(i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.000000</td>
<td>1.000000</td>
</tr>
<tr>
<td>0.1</td>
<td>1.105171</td>
<td>1.105171</td>
</tr>
<tr>
<td>0.2</td>
<td>1.221403</td>
<td>1.221403</td>
</tr>
<tr>
<td>0.3</td>
<td>1.349859</td>
<td>1.349859</td>
</tr>
<tr>
<td>0.4</td>
<td>1.491824</td>
<td>1.491825</td>
</tr>
<tr>
<td>0.5</td>
<td>1.648721</td>
<td>1.648721</td>
</tr>
<tr>
<td>0.6</td>
<td>1.822118</td>
<td>1.822119</td>
</tr>
<tr>
<td>0.7</td>
<td>2.013753</td>
<td>2.013753</td>
</tr>
<tr>
<td>0.8</td>
<td>2.225541</td>
<td>2.225541</td>
</tr>
<tr>
<td>0.9</td>
<td>2.459603</td>
<td>2.459603</td>
</tr>
<tr>
<td>1</td>
<td>2.718282</td>
<td>2.718282</td>
</tr>
<tr>
<td>1.1</td>
<td>3.004165</td>
<td>3.004166</td>
</tr>
<tr>
<td>1.2</td>
<td>3.320117</td>
<td>3.320117</td>
</tr>
<tr>
<td>1.3</td>
<td>3.669297</td>
<td>3.669297</td>
</tr>
<tr>
<td>1.4</td>
<td>4.055199</td>
<td>4.055200</td>
</tr>
<tr>
<td>1.5</td>
<td>4.481689</td>
<td>4.481689</td>
</tr>
<tr>
<td>1.6</td>
<td>4.953032</td>
<td>4.953032</td>
</tr>
<tr>
<td>1.7</td>
<td>5.473946</td>
<td>5.473947</td>
</tr>
<tr>
<td>1.8</td>
<td>6.049647</td>
<td>6.049647</td>
</tr>
<tr>
<td>1.9</td>
<td>6.685894</td>
<td>6.685894</td>
</tr>
<tr>
<td>2</td>
<td>7.389055</td>
<td>7.389056</td>
</tr>
</tbody>
</table>

We note that there are minor differences between the approximation computed by our function, and that computed by Maple's built-in version of the exponential function because we specified that our loop's tolerance should stop when the terms are $10^{-5}$ in magnitude, even though we are printing six decimal places.

Also note that we are free to use $i$ in the script even though we are also using $i$ in the function. Declaring $i$ to be local within the function means that its use within eApprox is insulated from the use of that variable outside of the function definition. If we had not declared $i$ to be local, the function's use of $i$ would clash with that of script's. This would break the use of $i$ by the script.
### 17.10 Troubleshooting local variable declarations

#### Table 17.13: Warning messages from undeclared local variables

<table>
<thead>
<tr>
<th>Warning messages from undeclared local variables</th>
<th>If you forget to declare a variable within a function definition as a local variable, then Maple will do it for you but print out a warning. While this default action is almost always what is wanted, the warning is get you to think about whether you really wanted s to be a global variable (the same values used within the procedure and outside of it) rather than local.</th>
</tr>
</thead>
<tbody>
<tr>
<td>eApprox2 := proc(x, tol) local term; s := 1; term := 1; for i from 1 while abs(term)&gt;=tol do term := term*x/i; s := s+term; end do; return s; end proc:</td>
<td>Warning, <code>s</code> is implicitly declared local to procedure <code>eApprox2</code></td>
</tr>
<tr>
<td>Warning, <code>i</code> is implicitly declared local to procedure <code>eApprox2</code></td>
<td>We see that s's value within this script is not affected to what happened to s within eApprox2. s was automatically declared local to the eApprox2 procedure when it was defined, even if we didn't include it in the code.</td>
</tr>
<tr>
<td></td>
<td>We can redefine eApprox2. This doesn't change the meaning but it does get rid of the warning messages.</td>
</tr>
<tr>
<td>Note that the computed result is based on i (the script version, not the local variable used inside eApprox2)’s last value from the previous computation in the worksheet. Since a for-loop variable's final value is the step beyond its limit, i's value at this point is 1.0 + .1 = 1.1.</td>
<td></td>
</tr>
</tbody>
</table>

```maple
eApprox2 := proc(x, tol)
  local term, s := 1, 1;
  for i from 1 while abs(term)>=tol do
    term := term*x/i;
    s := s+term;
  end do;
  return s;
end proc:
eApprox2 := proc(x, tol)
  local s, i, term;
  s := 1;
  term := 1;
  for i from 1 while abs(term)>=tol do
    term := term*x/i;
    s := s+term;
  end do;
  return s;
end proc:
```
17.11 Problem solving aided by function definition and repeated use

Table 17.14: A problem-solving situation with multiple use of a Maple procedure

(From Sullivan, Pre-calculus, p. 342)

A fast-food restaurant wants a special container to hold coffee. The restaurant wishes the container to quickly cool the coffee from 200 °F to 130 °F as quickly as possible, and keep the liquid between 110 °F and 130 °F (optimal drinking temperature) as long as possible. The restaurant has three containers to select from.

(a) The CentiKeeper Company has a container that reduces the temperature of a liquid from 200 °F to 100 °F in 30 minutes by maintaining a constant temperature of 70 °F.

(b) The Temp Control Company has a container that reduces the temperature of a liquid from 200 °F to 100 °F in 25 minutes by maintaining a constant temperature of 60 °F.

(c) The Hot'n'Cold Company has a container that reduces the temperature of a liquid from 200 to 120 °F in 20 minutes by maintaining a constant temperature of 65 °F.

How long does it take each container to lower the coffee temperature from 200 °F to 130 °F?

How long will the coffee temperature remain between 110 °F and 130 °F?

We created the following script to do problem (a)

Table 17.15: A script to solve problem (a) that doesn't define functions or procedures

\[
x(t) = T + (u_0 - T) e^{-kt}
\]

We present the code in Maple

\[
eApprox2(t, 10^2(-5), \exp(t));
\]

\[
\begin{align*}
T & = 3.004165230 \quad \text{(17.36)} \\
n & = 1.1 \quad \text{(17.37)}
\end{align*}
\]
\[ u(30) = 70 + 130 e^{30k} \]  
(17.43)

\[ 100 = 70 + 130 e^{30k} \]  
(17.44)

\[ f_{solve}(1.11.7), k \]  
\[-0.0488790229\]  
(17.45)

\[ u(t) = 70 + 130 e^{-0.0488790229t} \]  
(17.46)

Time to fall to 130 degrees.

\[ f_{solve}(rhs((1.11.9)) = 130) \]  
15.81880261  
(17.47)

Time to fall to 110 degrees:

\[ f_{solve}(rhs((1.11.9)) = 110) \]  
24.11427130  
(17.48)

Time to stay between 110 and 130

(1.11.11)−(1.11.10)

8.29546869  
(17.49)
To turn our "coffee script" into a procedure, we will need to do the same thing as before: identify the variables used as inputs, which variables other than the inputs are used in the script as local variables, and what the results are. We also need to choose a name for the procedure. Most of these findings are highly similar to those of the previous examples and we do not discuss them further. However, there is one difference: there are actually two results computed by the script, the time within the target range of temperatures, and the plot of the temperature graph. Since a function (whose rules of behavior a Maple procedure must follow) returns only one result, we put the two items into a list, and return a list as the result of the procedure.

Table 17.16: Coffee script as a Maple procedure

```maple
coffeeSolution := proc(constTemp, initPeriod, initTemp, nextTemp)
    local NewtonEquation, k, t, kval, initCondEq,
          keqn, teqn, texpr, timeInTarget,
          tStart, tEnd, pResult;
    NewtonEquation := u(t) = T+(u[0]-T)*exp(k*t);
    #Determine the value of the constant k in Newton's equation from
    #the data provided by the parameters.
    initCondEq := eval(NewtonEquation, [T = constTemp, t = initPeriod, u[0] = initTemp]);
    keqn := eval(initCondEq, u(initPeriod) = nextTemp);
    kval := fsolve(keqn, k);
    #Set up equation in t with the computed value of the heat constant.
    teqn := eval(NewtonEquation, [T = constTemp, u[0] = initTemp, k = kval]);
    texpr := rhs(teqn);
    # Compute
    tStart := fsolve(texpr=initTemp,t);
    tEnd := fsolve(texpr=nextTemp,t);
    # Time to stay within initTemp and nextTemp degrees
    timeInTarget := tEnd - tStart;
    pResult := plot(texpr, t = 0 .. 40, labels = ["time", "temperature in F"]);
    return ([timeInTarget, pResult]);
end proc;
```
end proc:

#Get the function to do the computation.
resulta := coffeeSolution(70, 30, 200, 100);

#Print out the pieces
printf("Time within target range for (a): \%f", result[1]);
print(result[2]);

Time within target range for (a): 30.000000

Once we have this function working, we can reuse it to compute solutions to parts (b) and (c) in the problem

Table 17.17: Reuse of the Maple procedure to solve parts (b) and (c) of the problem

<table>
<thead>
<tr>
<th>Reuse of the Maple procedure to solve parts (b) and (c) of the problem</th>
</tr>
</thead>
</table>
| #Get the function to do the computation.
resultb := coffeeSolution(60, 25, 200, 100);

#Print out the pieces
printf("Time within target range for (b): \%f", resultb[1]);
print(resultb[2]); |
| 17.11 Problem solving aided by function definition and repeated use • 267 |
print(resultc[2]);

[25.00000000, \textit{PLOT}(...)]

\textbf{Time within target range for (b): 25.000000}


[20.00000000, \textit{PLOT}(...)]

\textbf{Time within target range for (b): 20.000000}
17.12 Summary of the chapter

1. Functions with longer, more complex coding can be defined in Maple with the use of \texttt{proc... end proc;}

<table>
<thead>
<tr>
<th>General form of a Maple procedure definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>\texttt{function name := proc (formal parameters) }</td>
</tr>
<tr>
<td>\texttt{local variables used within procedure; }</td>
</tr>
<tr>
<td>\texttt{statements; }</td>
</tr>
<tr>
<td>\texttt{return expression or variable containing result of the function; }</td>
</tr>
<tr>
<td>\texttt{end proc; }</td>
</tr>
</tbody>
</table>

Notes

1. There is no semi-colon between the ( ) containing the formal parameters and the word \texttt{local}. If there are no local variables for the procedure, then you need to put in the semi-colon: \texttt{proc (formal parameters) ; statements; } ....

2. The opposite of a "local variable" is a "global variable". We don't use them much in the Maple procedures we will be working with. You can read more about them if you look up "global" in Maple on-line help. There is an example of the declaration and use of a global variable here (Brian: provide hyperlink to reference below using global variable).

<table>
<thead>
<tr>
<th>A simple function created with \texttt{-&gt;} and with \texttt{proc}</th>
</tr>
</thead>
<tbody>
<tr>
<td>\texttt{f := (a, b) \rightarrow \frac{\sqrt{a \cdot b}}{2} }</td>
</tr>
<tr>
<td>\texttt{(a, b) \rightarrow \frac{1}{2} \sqrt{a \cdot b} }</td>
</tr>
<tr>
<td>\texttt{f(3.0, Pi) }</td>
</tr>
<tr>
<td>\texttt{0.8660254040 \sqrt{\pi} }</td>
</tr>
</tbody>
</table>

| \texttt{f2 := proc(a, b); } |
| \texttt{return \frac{\sqrt{a \cdot b}}{2}; } |
| \texttt{end } |
| \texttt{proc(a, b) return 1/2*sqrt(a*b) end proc } | (17.54) |
| \texttt{f2(3.0, Pi) } |
| \texttt{0.8660254040 \sqrt{\pi} } | (17.55) |

<table>
<thead>
<tr>
<th>A longer procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td>\texttt{#This function computes an approximation to exp(x) }</td>
</tr>
<tr>
<td>\texttt{#by summing terms until the next term computed }</td>
</tr>
<tr>
<td>\texttt{#is less than tol in absolute value. }</td>
</tr>
<tr>
<td>\texttt{eApprox := proc(x, tol) }</td>
</tr>
<tr>
<td>\texttt{local s, i, term; }</td>
</tr>
<tr>
<td>\texttt{s := 1; }</td>
</tr>
<tr>
<td>\texttt{term := 1; }</td>
</tr>
<tr>
<td>\texttt{for i from 1 while abs(term) &gt;= tol do }</td>
</tr>
<tr>
<td>\texttt{term := term \cdot x / i; }</td>
</tr>
<tr>
<td>\texttt{s := s + term; }</td>
</tr>
<tr>
<td>\texttt{end do; }</td>
</tr>
<tr>
<td>\texttt{return s; }</td>
</tr>
</tbody>
</table>
end proc;

\[ \text{proc}(x, tol) \]
\[ \text{local } s, i, \text{term}, \]
\[ s := 1; \text{term} := 1; \text{for } i \text{ while } \text{abs}(\text{term}) \leq \text{tol} \text{ do } \text{term} := \text{term}*x/i; s := s + \text{term} \text{ end do}; \]
\[ \text{return } s \]
\[ \text{end proc} \]

2. Functions can be coded to immediately halt and print out an error message if the programming detects something that is a mistake -- typically incorrect input to a function, or calculated result that is a mistake due to a programmer error in coding.

Function definition with error signaling

\texttt{#Returns the value of a fast or slow speed for a fan based on the fan state, which should be either the symbol high or low.}
\texttt{setSpeed := proc (fanState)}
\texttt{local speedFast, speedSlow;}
\texttt{speedFast := 5000; speedSlow := 2000;}
\texttt{if fanState = high then return speedFast}
\texttt{elif fanState = low then return speedSlow}
\texttt{else error "fanState is", fanState, "should be high or low" end if}
\texttt{end proc;}

\texttt{setSpeed(high); setSpeed(low); setSpeed(middle);}

\texttt{proc(fanState)}
\texttt{local speedFast, speedSlow;}
\texttt{speedFast := 5000; speedSlow := 2000;}
\texttt{if fanState = high then return speedFast}
\texttt{elif fanState = low then return speedSlow}
\texttt{else error "fanState is", fanState, "should be high or low" end if}
\texttt{end proc}

5000
2000

Error, (in setSpeed) fanState is, middle, should be high or low

When we build this into a function, we add a little error-detection into the function itself, so that it will abort operation of whatever code it is running in if it ever encounters a fanState that is neither high nor low.

We have designed a function that gives an error message that is intelligible by design.
18 Creating user interfaces with Maple Components

18.1 Chapter Overview

It is generally recognized that graphical user interfaces (GUIs) make it easier to input certain kinds of values (typically, numbers or selections from a small number of alternatives), and to present information in a more readily grasped format.

We discuss how to set up simple GUIs in Maple using the system’s pre-designed Maple Components, sometimes referred to as widgets. In using a Maple Component, you are getting a graphical component such as a slider, a button, or a gauge that can be configured and programmed to provide the desired GUI functionality in connection with your application.

18.2 Introduction

We have promoted the use of scripts and procedures as a way of facilitating reuse of code. The textual form of instructions has gained preeminence as the way of specifying the instructions in scripts and procedures because of the flexibility and power of text editing tools for creating and modifying code.

Programming a script is different from executing it repeatedly. Invoking a Maple procedure or a script can often require specification of many parameter values. The task of entering parameter values through typing can be error prone because -- users can always type the wrong key by mistake. Furthermore, looking at written output can be a slow way to understand the results of a computation. One graph can be worth a thousand lines of numbers or a long formula -- as the old adage goes, even in computation one picture is worth a thousand words.

Anyone who has used a commercial web site or desktop office application is aware that there are many other ways of getting information in or out: text fields, buttons, sliders, graphs, dials, etc. We show below some of the alternatives built into Maple. These built-in alternatives are referred to as Maple Components. The colloquial terminology for them, however, is widget.

<table>
<thead>
<tr>
<th>Table 18.1: An assortment of input widgets</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Slider" /></td>
</tr>
<tr>
<td><img src="image" alt="Rotary dial" /></td>
</tr>
</tbody>
</table>

Clicking on this will display a list of options that you can select from using the mouse.

In this chapter we explain how to set up simple GUIs in Maple by connecting some of these pre-built widgets to Maple procedures you have written. Operating an input widget (e.g., moving a slider) causes it to invoke a Maple procedure you have placed in the worksheet. The programming of the procedure typically includes use of built-in procedures that will alter output widgets (e.g. change the level of a gauge). We explain the details of how to get this to happen in the following sections of this chapter.

### 18.3 Placing a Maple Component in a worksheet

To place a component on a worksheet, open up the Components panel of the Palette on the left hand side of the Maple window, shown below. Drag the desired component (for example, the slider), to the desired location on the worksheet.

#### Table 18.2: An assortment of output widgets

<table>
<thead>
<tr>
<th>An assortment of Maplet output widgets</th>
</tr>
</thead>
<tbody>
<tr>
<td>Button and Plot Area</td>
</tr>
<tr>
<td>![Image of a button and plot area]</td>
</tr>
<tr>
<td>Meter</td>
</tr>
<tr>
<td>![Image of a meter]</td>
</tr>
<tr>
<td>Volume gauge</td>
</tr>
<tr>
<td>![Image of a volume gauge]</td>
</tr>
</tbody>
</table>

![Image of a dodecahedron]

Click on me to display dodecahedron.
Table 18.3: The Component Palette in Maple

You can separate multiple components on a line by typing spaces between them, or place them on separate lines by typing enter (return on some keyboards). A more precise alignment may be achieved by inserting a table (through Insert->Table and selecting the number of rows and columns you want) and putting components in the desired positions. Document tables can be a mixture of text and Maple calculations as well as components.

18.4 Configuration of Maple Components: an overview

Maple Components typically need to be configured before they are usable. Widget configuration includes how a widget looks. Examples of configuring widget appearance include the label that appears on a button (presumably something more indicative of purpose than the default, which is just "Button"), the range of values that a slider supports, or whether a text field component will let the user edit it once the script/program using the widget begins execution.

Once a component is placed in the worksheet (as previously explained in section 18.3), it can be configured by right-clicking on it. A window will pop-up which allows specification of each settable component property. After configuration is complete, click on the OK button to exit configuration. We show an example of the widget configuration window in the example in section 18.6.

Every component has a name, which is one of the properties that can be set within the configuration window. Choosing a suggestive name allows one to more easily program applications where there are multiple components of the same type. For example, if a particular slider is to be used to specify the starting outdoor air temperature of an HVAC simulation, then it might be named outdoorTemp or something similar.

Another aspect of widget configuration is the programming that specifies what the widget does. It is written in a simple "mini-language" that lists an equation that includes widget names with a "%" prefixed to their names. This kind of programming is discussed in the next section.
18.5 Programming Maple Components

The programming of Maple widgets comes into action whenever the user clicks on, types into, or drags within an input widget.

At that point, the programming entered into the widget when it was configured is executed. We show an example of how to enter the programming in the example in the next section. In this section we describe the general form of the programming for widgets.

While any ordinary Maple programming (e.g. assignments, if, for, etc.) is allowed in widget programming, there is also a special kind of programming that can be done with the help of the Do operation of the DocumentTools package. Typically the programming includes one or more lines of the form

\[
\text{Do ( } \%\text{widgetName} = \text{expression} );
\]

This "Do" operation, which programs widgets, is different from the do keyword that we saw in Maple for and while loops. Typically, Do does not involve any repetition, which is a significant way it differs from do.

widgetName is the name of an output widget, as set up in that widget's configuration window. The expression can also include widget names with a % prefix. Anything in the expression that does not begin with a % is taken as ordinary Maple.

Typically, widget names given in the expression are input widgets. The values of the input widgets (e.g. the number a slider is set to) are used in the expression to compute a result. The result is then displayed by the output widget named widgetName listed on the left hand side of the equation.
18.6 An example of programming Maple Components

We're going to write an application that will plot sin(k*x), x = 0..10, where k is a value determined by a slider, and the color of the plot is determined from a combo box. The plot will be triggered whenever we click on the plot window. When it is working, it will look like this:

Table 18.4: A simple application to plot a curve specified by GUI widgets

<table>
<thead>
<tr>
<th>Ploting sin(k·x) for x between 0 and 10, for a specified color</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Directions</strong> operate the slider to select a value of k. Select a color from the combo box. Then hit the button and a plot of sin(k·x) will be drawn.</td>
</tr>
</tbody>
</table>

Drag a slider, a text area, a combo box, a button, and an embedded plot area into the table from the Components Palette.

Add descriptive text to the left of the text area and the combo box. This can be done in the same way as if you were to enter text into the table without any widgets around: position the cursor where you want to enter text, enter text mode by Insert->Text or control-T (command-T on Macintosh), and then use the keyboard.

2. By right-clicking, open up the configuration box of the slider and set it to allow integer values 1 to 10. Name the widget **SliderK**. Set Major Tick marks to 5, and minor tick marks to 1. Check "show axis labels", and "update continuously", as shown:
Close the configuration box by clicking on the "OK" button.

3. Right-click on the text area widget and open its configuration box. Configure its name, number of visible rows, and make it not editable. We set the latter property because we want to force the user to use the slider to set the value of $k$. 

---

Table 18.5: Slider configuration

<table>
<thead>
<tr>
<th>Option</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Name</td>
<td>SliderK</td>
</tr>
<tr>
<td>Tooltip</td>
<td></td>
</tr>
<tr>
<td>Value at Lowest Position</td>
<td>0</td>
</tr>
<tr>
<td>Value at Highest Position</td>
<td>10</td>
</tr>
<tr>
<td>Current Position</td>
<td></td>
</tr>
<tr>
<td>Spacing of Major Tick Marks</td>
<td>5</td>
</tr>
<tr>
<td>Spacing of Minor Tick Marks</td>
<td>1</td>
</tr>
<tr>
<td>Action When Value Changes</td>
<td>Edit</td>
</tr>
<tr>
<td>Options</td>
<td></td>
</tr>
<tr>
<td>Enable Input</td>
<td>✔</td>
</tr>
<tr>
<td>Visible</td>
<td>✔</td>
</tr>
<tr>
<td>Show Track</td>
<td>✔</td>
</tr>
<tr>
<td>Orient Vertically</td>
<td></td>
</tr>
<tr>
<td>Show Axis Labels</td>
<td>✔</td>
</tr>
<tr>
<td>Show Axis Tick Marks</td>
<td>✔</td>
</tr>
<tr>
<td>Snap to Axis Tick Marks</td>
<td></td>
</tr>
<tr>
<td>Update Continuously whileDragging</td>
<td>✔</td>
</tr>
</tbody>
</table>
Close the configuration box of the text area by clicking "OK".

4. Configure the combo box to allow the options "Choose color", "red", "green", and "blue". Name the widget "colorBox".

5. Configure the plot area widget so that its name is \textit{sinPlotter}.

6. We will now program the slider and the button so that when operated they affect the text area and plot area, respectively.

Right-click again on the slider, and select "Configuration Properties". When the configuration box appears, click on the "Edit...."
button for "Action When Contents Change". This brings up a code edit box. Add the line `Do( %kText = %SliderK)` to the window, so that it looks like this:

**Table 18.7: Programming of Slider**

```maple
use DocumentTools in
# Enter Maple commands to be executed when the specified
# action is carried out on the component.
# Use:
#   Do( %component_name );
# and
#   Do( %component_name = value );
# to set and get properties of the component.
# You can also use arbitrary expressions
# involving components, e.g.:
#   Do( %target = %input1 + 2*%input2 );
# Note the %-prefix to each component name.
# See ?CustomizingComponents for more information.

Do( %kText = %SliderK)

end use;
```

Most of the code that appears in the box are explanatory comments for the programmer. To summarize them, they tell you to add a `Do(....)` expression in the space provided between the `use DocumentTools in ...` and the `end use` statement.

`use ... end use` does a `with` with temporary rather than permanent effect. Recall from Chapter 8 that a `with` makes all the operations of a package available through their short name. `use DocumentTools ... end use` makes it possible to refer to the operation `DocumentTools[Do]` as just `Do` within the region of code delimited by the `use ... end use`.

Click on "OK" for the code edit box, and then again on "OK" in the configuration box to complete the programming of the slider.

7. In a similar fashion, program the `Draw Plot` button so that its code edit box looks like this:
8. You should now be able to operate the widgets -- select a value of k, select a color, and then hit the *Draw Plot* button to draw the plot.

18.7 Troubleshooting Maple Components

In this section we discuss problems typically encountered in setting up Maple components: syntax errors with "Do" programming, output widgets don't show the desired output, and input widgets need to be easier to set.

Syntax errors in Component code

The code edit box of Maple components has a "check now" button that can be operated before you close the programming window. This will check the syntax of the widget action programming and display a "Parse Error" dialog box if any errors are found. as illustrated in the "Code edit box with syntax error" table, below. Furthermore, if the programming window has "check syntax before saving", the window cannot be closed until all syntax errors are fixed in the window. We recommend that you do fix all syntax errors before closing the code edit box, since that will simplify subsequent attempts to debug the code. Of course, guaranteeing that the programming is free of typos does not mean that it will work properly, if there's still a difference between what you say and what you *mean* to say.
Table 18.9: Code edit box with syntax error

| Component code region with a syntax error in it. The programmer has mistakenly used `:=` rather than `=` in the attempt to cause the plot to be displayed in the GUI widget sinPlotter. | This error message appears if you click on the OK button or the Check Now button of the code region box. |

**Symptom: proper output does not appear in an output widget.**

The programming for widgets is a combination of ordinary Maple programming and the `%` and equation programming that references
the value or output of a widget. Double check what you are writing until you become expert on this variant of ordinary Maple programming. See the example in the following table

**Table 18.10: Example of GUI widget programming with subtle mistake**

<table>
<thead>
<tr>
<th>Example of GUI widget programming with subtle mistake</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="GUI widget programming code" /></td>
</tr>
</tbody>
</table>

The mistake is that the programmer used `sinPlotter` rather than `%sinPlotter`. Doing this produces no error message, but the plot area named `sinPlotter` does not display any plots as a result of this line of code. Editing the window so that the line of code says `Do (%sinPlotter = ...)` fixes the problem.

**Ambiguous settings**

Input widgets such as sliders or dials have the drawback that if there are many possible values it can be unclear to the user which value was selected. This is not a bug in programming, but it may cause malfunction of the application. For this reason, it can be good to set up a label or text area that displays the numerical value of the selection to supplement the reading of the dial or gauge, or to consider alternative tick mark and labeling.
Table 18.11: Three ways of reading the setting of a dial

<table>
<thead>
<tr>
<th>Three ways of reading the setting of a dial</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Dial Image 1" /></td>
<td>This dial is set to an integer value, but is it 27, 28 or 29?</td>
</tr>
<tr>
<td><img src="image2.png" alt="Dial Image 2" /></td>
<td>It's unambiguous what this dial is set to. It may not be that easy to set the dial to a particular value, but at least we know what the setting is.</td>
</tr>
<tr>
<td><img src="image3.png" alt="Dial Image 3" /></td>
<td>We have configured this dial in a way so that it's clear which of the possible settings was selected. Each setting has its own tick mark, and it's not too hard to figure out the value even if not every tick mark has a label. We couldn't do this very easily for a dial that needed to support 160 different settings.</td>
</tr>
</tbody>
</table>

18.8 Programming of Maple Components contrasted with other kinds of programming

Specialized mini-languages

Widget programming is simple but uses a special kind of syntax that works only with the DocumentTools package and with widgets. In this respect it is similar to the programming done within printf for formatted output -- %f, %e, \n, etc. are found only in strings used within printf (and its cousins, sprintf, and fprintf) and are a way of doing the mini-programming necessary for fine control of the output in Maple of numbers, strings, and other kinds of expressions.

The use of specialized mini-languages is a general phenomenon in computer systems. Such mini-languages tend to arise in situations where having a succinct notation with limited applicability that nevertheless leads to highly productive use of programmer time.

Other examples of mini-languages are the language used to format web pages, HTML (hypertext markup language -- see http://www.w3.org/MarkUp/Guide/), the language of regular expressions used to specify patterns in text search (http://www.regular-expressions.info/tutorial.html).

Such languages underscore the notion that proficient computer usage requires the willingness to learn several languages. Furthermore, learning languages is an on-going phenomenon: new languages will be invented and become popular as new types of problems are tackled on computers.

Programming by configuring icons

"Drag and configure to program" is a paradigm used in a number of other languages, such as the NXT language used to program Lego robots, or LabView (http://www.ni.com/labview/), a language used to control and process data from devices in a laboratory. These languages differ from Maple Component programming in that a) the objects found on the screen with NXT or LabView
programming stand and control other things such as wheels, voltage meters, light switches, etc whereas the programming of Maple components controls the component seen in the worksheet. b) The programming diagrams in NXT or LabView includes not only the icons, but lines of control and flow between them. One can see the order in which things are executed in a NXT or LabView programming by following the connecting lines. With Maple, the connections between components are implicit rather than visual -- mentioning the name of a component in another component's code edit box establishes the cause-and-effect between components, but there are no graphical, diagrammatic connections between components.

Table 18.12: NXT programming

The yellow and green icons are individual instructions. Each icon can be clicked on to configure its properties and therefore affect the programming. This diagram describes a complete program for control of a robot. Execution begins at the left and follows the paths and arrows indicated in the diagram. In this case, the backward orange arrow at the time indicates that a looping occurs.
LabView is used to control devices. Connections between icons indicates the flow of data between lab devices that "emit" data and those that process, record or use data.

### 18.9 Final words on user interfaces

Interactive systems such as Matlab, Mathematica, and Python have similar techniques for quickly setting up GUls and connecting them to computational code. There are some languages such as Tcl/Tk, Java, or C# where custom-built GUI widgets can be implemented. This allows designers to go beyond the limitations of whatever built-in components may provide. It may be possible to import a custom-made widget written in Java into one of the interactive systems and thus get both customizability and lots of pre-defined mathematical functionality.

In some applications, the program designer envisions that all the interaction will be through the GUI widgets. Interaction with the normal Maple worksheet may even be disadvantageous if the designer of the worksheet computation thinks that the typical user is might make keyboard entry mistakes with Maple syntax or the entry of parameter values that would cause the script or program to malfunction. For these kinds of situations, Maple provides Maplets, which is a way of running Maple so that only a GUI is shown and it is the only way the user can interact with the program. It is also possible to have a Maplet encoded in a web page so that the user's computer need only connect to a web server that has Maple and need not run Maple itself. You can read more about Maplets in Maple's on-line help. A page of the on-line help that shows how to build a simple Maplet is reproduced at [http://www.maplesoft.com/support/help/Maple/view.aspx?path=examples/MapletBuilder](http://www.maplesoft.com/support/help/Maple/view.aspx?path=examples/MapletBuilder).
18.10 Summary of the chapter

To use a Maple Component as for GUI input or output:

a) Drag a copy of a component from the Components palette, to the desired location of the worksheet.
b) Right-click (control-click) on the component in the worksheet to open the configurations box.
c) Specify the configuration for the component. Name the widget something that suggests its purpose, rather than a generic name such as "Slider1". This will make it easier to remember names when you are using many widgets in an application.
d) Open and edit the code edit box to specify the actions that should occur when the user changes the state of the component. Typically, one affects other components by putting in an equation that mentions the widgets by name. The component programming distinguishes between ordinary Maple variables and functions, and widgets by preceding widget names with a %.

<table>
<thead>
<tr>
<th>Slider configuration</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Name:</strong> Slider1</td>
</tr>
<tr>
<td><strong>ToolTip:</strong></td>
</tr>
<tr>
<td><strong>Value at Lowest Position:</strong> 1</td>
</tr>
<tr>
<td><strong>Value at Highest Position:</strong> 10</td>
</tr>
<tr>
<td><strong>Current Position:</strong> 5</td>
</tr>
<tr>
<td><strong>Spacing of Major Tick Marks:</strong> 1</td>
</tr>
<tr>
<td><strong>Spacing of Minor Tick Marks:</strong> 1</td>
</tr>
<tr>
<td><strong>Action When Value Changes:</strong> Edit...</td>
</tr>
<tr>
<td><strong>Options:</strong></td>
</tr>
<tr>
<td>□ Enable Input</td>
</tr>
<tr>
<td>□ Visible</td>
</tr>
<tr>
<td>□ Show Track</td>
</tr>
<tr>
<td>□ Orient Vertically</td>
</tr>
<tr>
<td>□ Show Axis Labels</td>
</tr>
<tr>
<td>□ Show Axis Tick Marks</td>
</tr>
<tr>
<td>□ Snap to Axis Tick Marks</td>
</tr>
<tr>
<td>□ Update Continuously while Dragging</td>
</tr>
</tbody>
</table>

[Image of slider configuration dialog box]
Programming of Slider

use DocumentTools in
# Enter Maple commands to be executed when the specified
# action is carried out on the component.
# Use:
#    Do( %component_name );
# and
#    Do( %component_name = value );
# to set and get properties of the component.
# You can also use arbitrary expressions
# involving components, e.g.:
#    Do( %Target = %Input1 + 2*%Input2 );
# Note the %-prefix to each component name.
# See TCustomizingComponents for more information.

Do( %Text = %SliderK )

end use;
19 Calculus and optimization

19.1 Chapter Overview

One of the most distinctive features of Maple is its ability to do mixed computations where the results are a mixture of formulas and numbers. We introduce some of the built-in functions for doing calculus: differentiation, limits, and finding minima and maxima. Like solve, the minima and maxima-finders come in two styles -- one that tries to calculate formulae for answers (exact calculation) and one that uses approximation techniques that work only when the answer is numeric.

19.2 Learning about additional kinds of mathematical computations

We have become accustomed to looking at problems that can be attacked by using some combination of solve, plot (and other visualization tools), and numerical evaluation/assignment. Contrasting this with the kind of calculation available from today's calculators, we observe that it is different qualitatively from the calculation provided by a calculator because

a) Programming (e.g. looping) allows us to do much larger quantities of calculation than we could through operations only through keystroke.

b) Programming (procedures and scripts) allows us to more easily reuse calculations on varieties of similar problems.

c) The document interface allows us to record the results and commentary, for future reference and/or use in presentations and reports.

d) Powerful operations such as solve make solving certain types of problems much more convenient, even with the work to needed to enter the expression into the computer.

e) Powerful visualization and animation features available on a computer enhance understanding of the phenomena being studied.

In addition to programmability and a more extensive user interface, computers in 2011 have another edge over calculators -- they can conveniently access much more library programming because they have the hard drive space and interface flexibility. Systems such as Maple, Mathematica and Matlab use this room to provide thousands of extra mathematical features. Some of it is immediately useful to first year university/college students in the form of symbolic computation -- results from calculus. Some of it involves operations and functions that you will hear about after the first year -- linear algebra, multivariate calculus, ordinary and partial differential equations, Laplace transforms, etc. Learning how to operate a system such as Maple on the basic features gives you the understanding to pick up many of the more advanced features on your own, after you learn the mathematics. We give you a taste of this by exploring some of operations that are comprehensible with first year mathematics.

However, no course could possible teach all the features needed in a career. Taking advantage of systems with thousands of features means users must become self-teaching after their initial period of training. The process is still the same -- using a combination of written documentation and interaction with more experienced users. However, no matter how good the support system is, there will always be gaps that the user will have to fill in, through thought and experimentation, on their own. It's work to do this, but the payoff is that you are freed from only using the features you saw in a course.

19.3 Differentiation, simplification

As it is taught in traditional first-year calculus, differentiation is an operation on functions. Maple knows how to differentiate all of the common functions found in calculus. It is particularly useful for performing differentiation when it would take a lot of algebraic manipulation to do the operations by hand.

\texttt{diff} as a function takes two or more arguments. The first argument must be (or evaluate to) an expression. The second argument must be (or be an expression that evaluates to) the variable of differentiation. The there are third, fourth, etc. arguments provided, they are used as variables of higher derivatives.

The result of \texttt{diff} is an expression or a number if the derivative is a numerical constant.

Evaluation of a derivative expression can occur through \texttt{eval} as with other expressions.
Table 19.1: Symbolic differentiation using the expression palette

<table>
<thead>
<tr>
<th>Expression</th>
<th>Derivative w.r.t.</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 - 2x + 5$</td>
<td>differentiate w.r.t. $x$</td>
<td>$2x - 2$</td>
</tr>
<tr>
<td>$\sin(\omega t + 5)^2$</td>
<td>differentiate w.r.t. $t$</td>
<td>$2\omega \sin(\omega t + 5) \cos(\omega t + 5)$</td>
</tr>
<tr>
<td>simplify symbolic</td>
<td></td>
<td>$2\cos(\omega t + 5)^2 \omega^2 - 2\sin(\omega t + 5)^2 \omega^2$</td>
</tr>
<tr>
<td>$\frac{d}{d\omega} \sin(\omega t + 5)^2$</td>
<td></td>
<td>$2\sin(\omega t + 5) \cos(\omega t + 5) t$</td>
</tr>
</tbody>
</table>

Table 19.2: Symbolic differentiation and evaluation of derivatives, textually

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code> expr := x^2 - 2x + 5</code></td>
<td>Find the first derivative of the expression with respect to $x$.</td>
</tr>
<tr>
<td><code>diff(expr, x)</code></td>
<td>$2x - 2$ (19.2)</td>
</tr>
<tr>
<td><code>postExpr := sin(\omega t + 5)^2</code></td>
<td>Find the first derivative of the expression with respect to $t$.</td>
</tr>
<tr>
<td><code>diff(postExpr, t)</code></td>
<td>$2\sin(\omega t + 5) \cos(\omega t + 5) \omega$ (19.4)</td>
</tr>
<tr>
<td><code>diff(postExpr, t, t)</code></td>
<td>$2\cos(\omega t + 5)^2 \omega^2 - 2\sin(\omega t + 5)^2 \omega^2$ (19.5)</td>
</tr>
<tr>
<td><code>diff(expr, t)</code></td>
<td>$0$ (19.6)</td>
</tr>
<tr>
<td><code>simplify(2\cos(\omega t + 5)^2 \omega^2 - 2\sin(\omega t + 5)^2 \omega^2)</code></td>
<td><code>diff</code>'s second argument must be the variable of differentiation. Note that if the variable doesn't occur in the expression, the derivative (according to the mathematical definition) is zero. If you are surprised by getting a zero derivative, check that you are using the correct variable.</td>
</tr>
<tr>
<td><code>eval(diff(x^2 - 2x + 5, x, x = 3))</code></td>
<td>This is a way to compute $\frac{d}{dx} x^2 - 2x + 5 \bigg</td>
</tr>
<tr>
<td><code>eval(diff(sin(omega*t + 5)^2, t, t, t = 47.0))</code></td>
<td>This is a way to compute $\frac{d^2}{dt^2} \sin(\omega t + 5)^2 \bigg</td>
</tr>
</tbody>
</table>
In the second derivative, but we only evaluated one of them, \( t \).

**Table 19.3: Plotting a function and its derivative together**

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f := (a, b) \rightarrow \sin(a) + \cos(b) ) ( (a, b) \rightarrow \sin(a) + \cos(b) ) (19.10)</td>
<td>We're already used to functions that take one or two arguments.</td>
</tr>
<tr>
<td>( g := (\alpha) \rightarrow f\left(\frac{\alpha}{2}, \frac{\alpha}{3}\right) ) ( \alpha \rightarrow f\left(\frac{1}{2} \alpha, \frac{1}{3} \alpha\right) ) (19.11)</td>
<td>In this example, we plot a &quot;strange&quot; function built out of trigonometric parts, and plot it and its derivative. Since ( g ) is a function ( g(t) ) will evaluate to the expression. Thus the plotting variable should be ( t ) rather than ( \alpha ) or some other variable. Recall from Chapter 3 that we can get the character ( \alpha ) either by using the Greek Palette. Another way of getting ( \alpha ) is to type alpha; Maple will auto-transform it typographically.</td>
</tr>
<tr>
<td>( plot(g(t), t = 0..30) )</td>
<td></td>
</tr>
<tr>
<td>( fderiv := \text{diff}(g(\alpha), \alpha) ) ( \frac{1}{2} \cos\left(\frac{1}{2} \alpha\right) - \frac{1}{3} \sin\left(\frac{1}{3} \alpha\right) ) (19.12)</td>
<td>The value of ( fderiv ) is an expression involving alpha. In programming, unlike mathematics, variable names often have names that are more than one letter long. This is to increase intelligibility to readers looking at the programming. Although it seems minor, ease of comprehension can play an significant role in the cost of developing and using software, so is an important engineering concern.</td>
</tr>
</tbody>
</table>
We plot two expressions together involving the variable $\alpha$ on the same plot. The use of a set \{ \} as a first argument to plot to do multiple plots was first explained in section 6.3.

We need to use $\alpha$ as the plotting variable here since the value of $\text{fderiv}$ is an expression involving $\alpha$.

19.4 Limits

You can use the clickable interface to compute limits by selecting the appropriate item from the Expression palette and then filling in the template as needed. Maple uses calculus techniques (e.g. l'Hôpital's Rule) to compute limits symbolically.

Table 19.4: Clickable interface version of limits

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lim_{x \to 3} \frac{1}{(x - 3)^2}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>$\lim_{x \to -\infty} \frac{\sin(x)}{x}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\lim_{t \to 0} \frac{1}{t}$</td>
<td>undefined</td>
</tr>
<tr>
<td>$\lim_{t \to 0^-} \frac{1}{t}$</td>
<td>$-\infty$</td>
</tr>
<tr>
<td>$\lim_{x \to \alpha} \frac{\sin(x)}{x}$</td>
<td>$\frac{\sin(\alpha)}{\alpha}$</td>
</tr>
</tbody>
</table>

The result of limit can be an expression (possibly involving positive or negative infinity) as well as the symbol undefined.

Even though the limit exists as $t$ approaches from the right or left, they do not agree, so there is no "two-sided" limit.

To take a one-sided limit, add a "+" or "-" superscript to the limit point.

Note that the value returned as the limit for $x=\alpha$, while true for most values of $\alpha$, is not really valid for $\alpha=0$. 
The textual version of taking limits involves the `limit` function. `limit` takes at least two arguments. The first argument is the expression that you wish to take the limit of. The second argument is an equation indicating the limit variable and the limiting value. If you supply a third argument, it indicates whether a "right sided", "left sided" limit is desired instead of a two-sided limit.

Table 19.5: Textual version of limits

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>limit(1/(x-3)^2,x=3)</code></td>
<td>The first argument is the expression that you wish to take the limit of. The second argument is an equation indicating the limit variable and the limiting value.</td>
</tr>
<tr>
<td><code>limit(sin(x)/x,x=∞)</code></td>
<td>You can use &quot;infinity&quot; or &quot;-infinity&quot; as the way of specifying $\infty$ or $-\infty$ textually without use of the palette.</td>
</tr>
<tr>
<td><code>limit(1/t,t=0,left)</code></td>
<td>The third optional third argument to <code>limit</code> can specify a one sided limit. This is the textual way of specifying $\lim_{t \to 0^-} \frac{1}{t}$. There is no simple way of specifying this.</td>
</tr>
<tr>
<td><code>limit(1/t,t=0,right)</code></td>
<td>This is the textual way of specifying $\lim_{t \to 0^+} \frac{1}{t}$.</td>
</tr>
</tbody>
</table>

It would be nice if the second argument for the textual form of `limit` was `var -> limitValue` instead of `var = limitValue`, since arrows are the more conventional notation. But because of the limitations of Maple's processing capabilities for its programming language, it would be more expensive to support arrows for both this meaning in limits and the use of `->` in function definitions. So users must get used to using equations rather than the standard math symbol for "approaches".

19.5 Finding minima and maxima by using Maple as a calculus calculator

The minimum or maximum of a continuous function in an interval is sometimes referred to as the local extrema of the function. The function whose extremum you are finding is referred to as the objective function. In elementary calculus, you learn how to find extrema through the use of derivatives and a little algebra.

Maple can be used to do such calculations using `diff` to find the derivative, and then finding where the derivative is zero through `solve` or `fsolve`. The advantages of doing it with a system such as Maple are the same as with other calculations: a) it is easier to do if the formula is involved or if you don't remember all the math, b) you can easily organize the information about the problem and its solution using Maple word processing so that the work can be easier to present or read, c) the computer's speed of re-execution makes it easy to handle a bundle of similar problems.

Table 19.6: Finding the minimum and maximum using textually-specified commands

<table>
<thead>
<tr>
<th>From Anton, Calculus 8th ed. p. 307 problem 13 (modified)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Find the absolute maximum and minimum values of $f := (x) \rightarrow 2 - 2 \cdot \sin(x)$ for $x$ in $\pi$</td>
</tr>
</tbody>
</table>

Solution

<table>
<thead>
<tr>
<th>deriv := <code>diff(f(x),x)</code></th>
<th>We define a continuous function</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2 \cos(x)$</td>
<td>and compute its derivative symbolically.</td>
</tr>
</tbody>
</table>
We generate a plot to get an idea of where to look for minima and maxima. Evidently the minimum is in the interior of the interval middle and the maximum is at the left.

Finding where the derivative is zero should produce the minimum.

\[
\frac{1}{2} \pi
\]  

The minimum is confirmed to be zero -- the plot seems to indicate that but the picture doesn't let us know whether it's really zero or just really close to zero.

Standard calculus procedure is to check out the value of the end points, since the minimum might be there instead of in the interior of the interval. However, the values of \( f \) at both of the end points are larger than the value at \( \frac{\pi}{2} \). We conclude that \( f \) attains a minimum at \( x = \frac{\pi}{2} \).

Table 19.7: Writing a procedure to calculate minimum of a function and produce a plot that will help understanding of it

We enter the expression, and plot it on -2..3 using the 2-D plot builder.
To develop the script, first we do it interactively.

We do a chain of calculations to find what is obviously the minimum point.

We had to copy the expression again, and evaluate it at $x = -\frac{1}{2}$ to calculate the minimum value. Once we have a number, we calculate what it is, approximately.

We omit calculating the value of the objective expression at the end points because the plot tells us that minimum is not located there.

The next step is to develop the textual version of this.

We turn the sequence of actions into a textual script. The parameters are the expression to be minimized, the variable in the expression and the region. The plot is just to help us have a visual check that the numeric answer result is probably correct. Since it's something we decide we'd want to do with any version of a minimization problem, we decide to include it in the script.
We turn this into a procedure that does the same thing, returning the minimum value and its location in a list. Unless we print the plot, it won't appear during execution of the procedure because by default Maple only displays the result of the procedure. print causes something to be printed in addition to the final result.

```plaintext
plot(expr, var = interval)

deriv := diff(expr, var)
\[ \frac{2 \times 1}{3} \left( x^2 + x + 5 \right)^{1/3} \]  (19.29)

soln := solve(deriv, x)
\[-\frac{1}{2}\]  (19.30)

valAtPoint := eval(expr, var = soln)
\[ \frac{1}{4} 19^{2/3} 4^{1/3} \]  (19.31)

result := evalf(valAtPoint)
2.825719660  (19.32)

findMin := proc(expr, var, interval)
    local deriv, soln, valAtPoint;
    print(plot(expr, var=interval));
    deriv := diff(expr, var);
    soln := solve(deriv, var);
    valAtPoint := eval(expr, var=soln);
    return [evalf(valAtPoint), var=soln];
end;
```
We solve the minimization problem.

\[
\text{findMin} := \text{proc}(\text{expr}, \text{var}, \text{interval}) \\
\text{local} \ \text{deriv}, \text{soln}, \text{valAtPoint}; \\
\text{print}(\text{plot}(\text{expr}, \text{var} = \text{interval})); \\
\text{deriv} := \text{diff}(\text{expr}, \text{var}); \\
\text{soln} := \text{solve}(\text{deriv}, \text{var}); \\
\text{valAtPoint} := \text{eval}(\text{expr}, \text{var} = \text{soln}); \\
\text{return} \ [\text{evalf}(\text{valAtPoint}), \text{var} = \text{soln}] \\
\text{end proc}
\]

\[
\text{findMin}\left([x^2 + x + 5 + \frac{2}{3}, x = -2..3]\right)
\]

\[
\begin{bmatrix}
2.825719660, x = -\frac{1}{2}
\end{bmatrix}
\]
Now that we have a procedure, we can solve a similar problem: minimize \( \sqrt{x^2 + x + 3} \) for \( y \) in the interval \([-1, 4]\). We see that the plot supports the numerical calculation that the value of the expression at the minimum is approximately 1.65 at \( y = -\frac{1}{2} \).

\[
\text{findMin}(\sqrt{y^2 + y + 3}, y, -1..4)
\]

19.6 One-step extrema-finding using maximize and minimize

Maple's library provides both exact and approximate numerical functions that do the entire sequence of steps needed to find extrema. In this section, we discuss the Maple functions that provide exact solutions to such problems. The exact version automates the work that the user would have to do with \texttt{diff} and \texttt{solve}. The approximate version will provide usable information in situations where exact solution methods don't work.

The Maple function for finding maxima is, appropriately enough, called \texttt{maximize}. If \texttt{expr} is an expression involving a variable \texttt{x}, then \texttt{maximize(expr; x=range)} will produce maximum value of the expression for \texttt{x} in that range. If there is no maximum value, or if Maple can't find it, \texttt{NULL} is returned. Thus if \texttt{maximize} returns an answer, it should be "believable", but a result of \texttt{NULL} doesn't mean that there is no minimum. It may just mean that Maple was unable to find it. \texttt{infinity} or \texttt{-infinity} may be an answer if the function is unbounded within the range.

We have already seen \texttt{max} and \texttt{min}, which are also built-in Maple functions. However, they only work on lists, sets of values. They don't work at finding minima or maxima of expressions.

It is often a good idea when working on extrema problems to do a plot of the expression in question, so that you can some notion of where the maxima will be and what their values will be like.

### Table 19.8: minimize and maximize

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>\texttt{max}</td>
<td>maxima of a list, set of values.</td>
</tr>
<tr>
<td>\texttt{min}</td>
<td>minima of a list, set of values.</td>
</tr>
<tr>
<td>\texttt{maximize}</td>
<td>maximum value of an expression for a variable in a range.</td>
</tr>
<tr>
<td>\texttt{minimize}</td>
<td>minimum value of an expression for a variable in a range.</td>
</tr>
</tbody>
</table>

| minimize and maximize
<table>
<thead>
<tr>
<th>From Anton, Calculus 8th ed. p. 307 problem 13 (modified)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Find the absolute maximum and minimum values of</td>
</tr>
<tr>
<td>( f := (x) \rightarrow 2 - 2 \cdot \sin(x) ) for ( x ) in ( -\frac{\pi}{4}, \pi )</td>
</tr>
<tr>
<td>Solution</td>
</tr>
</tbody>
</table>
**minimize and maximize**

\[ f := (x) \rightarrow 2 - 2 \cdot \sin(x) \]

\[ x \rightarrow 2 - 2 \sin(x) \]  

(19.36)

We generate a plot to get an idea of where to look for minima and maxima. Evidently the minimum is in the middle and the maximum is at the left.

The minimum is confirmed to be zero -- the plot seems to indicate that but the picture doesn't let us know whether it's really zero or just really small. Note again that this is the minimum value, it is not the location of the minimum value.

\[ \text{smallestValue := minimize}\left[f(x),x=-\frac{\pi}{4}..\pi\right] \]

0  

(19.37)

Finding the maximum is a copy/edit job once we've worked out how to do the minimum.

\[ \text{largestValue := maximize}\left[f(x),x=-\frac{\pi}{4}..\pi\right] \]

\[ 2 + \sqrt{2} \]  

(19.38)

Sometimes, you want not only what the maximum or minimum value is, but also where it is. If you give `maximize` or `minimize` an extra third parameter, location, then it will return a sequence for a result. This sequence has a logical but somewhat intricate structure. You can use a chain of selections to extract the location \( \text{var=location point} \) from the answer given by Maple.

**Table 19.9: Solving a problem with minimize and maximize**

**Solving a problem with minimize and maximize**

From Anton, Calculus 8th ed. p. 307 problem 13 (modified)

Find the absolute maximum and minimum values of

\[ f := (x) \rightarrow 2 - 2 \cdot \sin(x) \text{ for } x \text{ in } -\frac{\pi}{4}..\pi \]

We define a function.

\[ f := (x) \rightarrow 2 - 2 \cdot \sin(x) \]

\[ x \rightarrow 2 - 2 \sin(x) \]  

(19.39)
Solving a problem with minimize and maximize

We generate a plot to get an idea of where to look for minima and maxima. Evidently the minimum is in the middle and the maximum is at the left.

We get the smallest value of \( f \) in the interval.

This is what minimize does with the extra parameter location. The result returned by minimize with this extra parameter contains not only the value of the minimum, but all the values of \( x \) where the minimum occurs. It takes a lengthy daisy chain of part-extraction operations to get the value out.

We use the parameter location to get all \( x \) values where the minimum occurs.
Solving a problem with minimize and maximize

\[
f(x) = 0 \quad (19.46)
\]

\[
x_{\text{Location}} := \text{minResult}(2 \mid 1 \mid 1)
\]

\[
\left\{ x = \frac{1}{2} \pi \right\} \quad (19.47)
\]

\[
\text{minValue} := \text{eval}(f(x), x_{\text{Location}})
\]

\[
0 \quad (19.48)
\]

\[
\text{maxResult} := \text{maximize}(f(x), x = -\frac{\pi}{4} \text{..} \pi, \text{location})
\]

\[
2 + \sqrt{2}, \left[ \left[ x = -\frac{\pi}{4}, 2 + \sqrt{2} \right] \right] \quad (19.49)
\]

\[
x_{\text{Location}} := \text{maxResult}(2 \mid 1 \mid 1)
\]

\[
\left\{ x = -\frac{\pi}{4} \right\} \quad (19.50)
\]

\[
\text{maxLocation} := \text{eval}(f(x), x_{\text{Location}})
\]

\[
2 + \sqrt{2} \quad (19.51)
\]

Table 19.10: Using optimization in a chemical production problem with minimize and maximize


A liquid form of penicillin manufactured by a pharmaceutical firm is sold in bulk at a price of $200 per unit. If the total production cost (in dollars) for \( x \) units is

\[
C(x) = 500000 + 80x + 0.003x^2
\]

and if the production capacity of the firm is at most 30,000 units in a specified time, how many units of penicillin must be manufactured and sold in that time to maximize the profit? What will the revenue be at the maximal profit?

**Solution**

\[
C := (x) \rightarrow 500000 + 80x + 0.003x^2
\]

\[
x \rightarrow 500000 + 80x + 0.003x^2 \quad (19.52)
\]

\[
R := (x) \rightarrow 200x
\]

\[
x \rightarrow 200x \quad (19.53)
\]

However, once we have figured it out, we can get the same result by daisy-chaining the operations together.

A copy/edit job will produce the similar sequence of operations for the maximum.

Finding the maximum is a copy/edit job once we've worked out how to do the minimum.
We plot the profit function to scope out its behavior. Evidently the function does hit a maximum, roughly around \( x = 20000 \). If we got \( x = 35 \) as the answer from our maximum calculation in Maple, we'd wonder if we did things correctly.

We are getting floating point answers from Maple because some of the numbers in \( C \) are floating point. If we wanted an exact calculation, we would change the .003 into \( \frac{3}{1000} \).

We use the daisy-chaining of operations we saw in the previous example to extract the value of the \( x \) where the maximum is obtained. We use the \texttt{floor} function which rounds down to the nearest integer. This is the way to guarantee that we have a result that is a whole integer, without fractional part. The chemical process can't produce a fraction of a unit.

We used the clickable menu item "Numeric Formatting" (see for example the "A script that uses functional composition" example in Chapter 8) to reformat the result in terms of currency.
19.7 One-step approximations to extrema using Optimization\[Maximize\] and Optimization\[Minimize\]

Maple's \texttt{solve} and \texttt{fsolve} provide exact and approximate solutions. In a similar fashion, \texttt{maximize} and Optimization\[Maximize\], and \texttt{minimize} and Optimization\[Minimize\] provide exact and approximate solution to extrema problems. Optimization\[Maximize\] and Optimization\[Minimize\] only work when there is a numerical answer. They use only numerical, not analytical knowledge about functions. They produce a "best effort" approximation.

The clickable interface's "Optimization" operation invokes these approximate optimizers.

Optimization\[Maximize\] and Optimization\[Minimize\] always return a list. The first item of the list is the extreme value, and the second item in the list is a list of equations describing the location of the extreme value. As with the exact optimizing functions \texttt{maximize} and \texttt{minimize}, the second item can be used as a parameter to eval to evaluate an expression at the location of an extreme value.

While the inventiveness of numerical analysts make such approximation techniques usually reliable, they can be fooled and without warning produce a bad approximation. For that reason, the results of \texttt{fsolve}, \texttt{Maximize} and \texttt{Minimize} should be checked against what else is known about the properties of the actual solution. For example, by plugging the location of the approximate maximum into the objective function, does it really produce a value that is larger than any other value of the objective function that you can think of? Often times a plot makes this easy to tell.

Table 19.11: Finding an approximation to the minimum with Optimization\[Minimize\]

<table>
<thead>
<tr>
<th>Ant. Calculus 8th ed. problem 47, p. 321 (modified)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two particles, A and B, are in motion in the xy-plane. Their coordinates at each instant of time ( t ) ((t \geq 0)) are given by ( x_A = t, y_A = 2 \cdot t ). ( x_B = 1 - t, (t \geq 0) ) are given by ( x_B = 1 - t, y_B = 2 \cdot t ). Find the minimum distance between ( A ) and ( B ).</td>
</tr>
<tr>
<td>( x_A := t )</td>
</tr>
<tr>
<td>( y_A := 2 \cdot t )</td>
</tr>
<tr>
<td>( x_B := 1 - t )</td>
</tr>
<tr>
<td>( y_B := t )</td>
</tr>
<tr>
<td>( p1 := [x_A, y_A] )</td>
</tr>
<tr>
<td>( p2 := [x_B, y_B] )</td>
</tr>
</tbody>
</table>

| \( t \) |
| \( 2t \) |
| \( 1 - t \) |
| \( t \) |

\( p1 \) and \( p2 \) are lists of the x and y positions for each point. dist is the objective function. This is the Euclidean distance between the two points. We can compute the distance between \( p1 \) and \( p2 \) using it.
It's always good to have an independent way of evaluating the answer you get from a "solver". In this case, we can plot the function and from that see approximately where the answer ought to be. Our first plot does not show the minimum that clearly, so we do another plot that emphasizes the region of interest.
Using *with*, we can then say Minimize rather than Optimization[Minimize].

The exact location is \( t = 0.4 \), but the rounding error made during the operation of Minimize leads to only an approximation of the exact answer. This is typical.

According to the structure returned by this function, the location information is in the second element of the result.

We can find the location of particles A and B at \( t=0.4 \) by evaluating the expression \( p1 \) and \( p2 \) at that value of \( t \). Note that \( \text{minLocation} \) is a list consisting of an equation, which \( \text{eval} \) can use as its second argument.

This is a problem that Maple's exact solver can solve exactly. The numerical approximation produced by Optimization[Minimize] is in close agreement with the exact result.

This expression has a complicated expression for the exact minimum. We could try to approximate it using the right-click->approximatemenu, or we could give it to the approximate minimizer instead.
While the approximate solvers will probably fare well on most functions that have only a few extrema, they can be fooled if the function has a lot of extrema close to each other. In that case, it may return something that is an extremum in a small neighborhood around the point, but not if a larger interval is considered. This "tricky" example is from Cheney and Kincaid's textbook on numerical analysis, *Numerical mathematics and computing*, 6th edition, 2008 Thomson Brooks/Cole, p.626.

We get an answer from approximation technique. There are no error or warning messages. To check this answer, let's do a plot and see what the function looks like.

We see why an approximate solver might have problems -- there are lots of "local minima". x=.56 does not seem to be a true minimum for in the interval. We can draw a zoomed in plot and look closer:
It's clear now that $x=0.56$ is not the location of the minimum. $x$'s value should be about $x=-0.02$ and the minimum value is about $-1$. These approximate coordinates were found using the cursor positioning features described in Table 2.13.

Applying the exact minimizer to this problem produces a hard to understand result that takes much more time to compute than the approximate technique. Nevertheless, it is the true minimum in the interval. We ask $\text{evalf}$ to approximate it. That produces something that, while still approximate, agrees more with what the plot leads us to expect for the right answer.

The moral: numerical optimization typically produces results quickly but even the best can be fooled. You need to verify the results if the cost of making an error is high. We might be tempted to see if we can change the circumstances so that the objective function we work with is not so "squirrelly". 

```
minResult := minimize(objectiveExpression, x = -5..3, location)

\[\begin{align*}
\frac{1}{2809} & \cdot \text{RootOf}(2\cdot_2 + 2809 \cdot \cos(\_2), \\
-1.569678720)^2 + \sin(\text{RootOf}(2\cdot_2 + 2809 \cdot \cos(\_2), \\
+ 2809 \cdot \cos(\_2), -1.569678720)) \cdot \left[\begin{array}{c}
x \\
= \frac{1}{53} \cdot \text{RootOf}(2\cdot_2 + 2809 \cdot \cos(\_2), \\
-1.569678720) \\
\frac{1}{2809} \cdot \text{RootOf}(2\cdot_2 + 2809 \cdot \cos(\_2), \\
+ 2809 \cdot \cos(\_2), -1.569678720)^2 + \sin(\text{RootOf}(2\cdot_2 + 2809 \cdot \cos(\_2), \\
-1.569678720)))\right]\end{align*}\]
```

```
\text{evalf}([\text{minResult}])

\[\begin{bmatrix}
-0.9991222337, \\
-0.02961657963, \\
-0.9991222337
\end{bmatrix}\]
```
19.8 Multivariate operations in Maple

diff, minimize, solve, and the rest can handle problems and situations where there are multiple variables involved. Approximation methods (e.g., fsolve or Optimization[MAXIMIZE] often take more prominence with multivariate problems because of the difficulty and/or expense of finding succinct formulas that express the solution. We won't explore Maple's abilities to handle multivariate problems here, but they are there to be explored and used when you're ready for it.

Table 19.13: Multivariate operations in Maple

\[
\begin{align*}
\frac{\partial}{\partial x} \frac{\partial}{\partial x} \sin \left( \frac{x}{y} \right)^2 &= \frac{2 \cos \left( \frac{x}{y} \right)^2}{y^2} - \frac{2 \sin \left( \frac{x}{y} \right)^2}{y^2} \\
diff \left( \sin \left( \frac{x}{y} \right)^2, x, y \right) &= \frac{2 \cos \left( \frac{x}{y} \right)^2 x}{y^3} - \frac{2 \sin \left( \frac{x}{y} \right)^2 y}{y^3} - \frac{2 \sin \left( \frac{x}{y} \right) \cos \left( \frac{x}{y} \right)}{y^2} \\
eqs := \{ x + 2 y = \frac{5}{4}, x^2 + y^2 = 1 \} &\quad \text{Compute the second partial derivative with respect to x.} \\
\text{solve}(\eqs, \{x, y\}) &= \left\{ x = -\frac{1}{2} \text{RootOf} \left( 5 \_Z^2 - 20 \_Z + 9, \text{label} = \_L1641 \right) + \frac{5}{4}, y = \frac{1}{4} \text{RootOf} \left( 5 \_Z^2 - 20 \_Z + 9, \text{label} = \_L1641 \right) \} \\
eqsf := \{ x + 2 y = 1.25, x^2 + y^2 = 1 \} &\quad \text{The textual way of entering partial derivatives.} \\
\text{solve}(\eqsf, \{x, y\}) &= \{ x = 0.9916198487, y = 0.1291900756 \}, \{ x = -0.9916198487, y = 0.8708099244 \} \\
expr := \sqrt{\left( x^2 + \sin \left( y + \frac{1}{10} \right)^2 \right)} + 5 &\quad \text{We are interested in finding the points of intersection with the unit circle with center at (0,0), and the line defined by x+y=5. Doing it exactly gives an answer, but it's not easy to understand -- x and y can be either the positive or negative solution to a quadratic expression. Introducing a floating point number into the equation (1.25 instead 10/8) tells solve that it's all right to approximate the results.} \\
&\quad \text{We want to find the minimum value of an expression in two variables. minimize takes a long time to work and doesn't seem to come up with much. However, the approximate minimizer finds a value of 5.} 
\end{align*}
\]
minimize(expr, x=-1..1, y=-1..1)

Warning, computation interrupted

Optimization[Minimize](expr, x=-1..1, y=-1..1)

[5.00000000000000000, [x=0, y=
-0.100000000000000000]]

plot3d(expr, x=-1..1, y=-1..1, axes=boxed)

Multivariate plotting is handled by the plot3d function. By clicking on
the plot window, the point of view can be moved or rotated. Animations,
etc. can be built in a fashion similar to two dimensional plots.

19.9 Chapter summary

**Symbolic differentiation using the expression palette, right-click, control-\textasciitilde.**

<table>
<thead>
<tr>
<th>Expression</th>
<th>Differentiation</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 - 2x + 5 )</td>
<td>\text{differentiate w.r.t. } x</td>
<td>( 2x - 2 )</td>
</tr>
<tr>
<td>( \sin(\omega t + 5)^2 )</td>
<td>\text{differentiate w.r.t. } t</td>
<td>( 2 \sin(\omega t + 5) \cos(\omega t + 5) \omega )</td>
</tr>
<tr>
<td>( \text{simplify symbolic} )</td>
<td></td>
<td>( 2 \cos(\omega t + 5)^2 \omega^2 - 2 \sin(\omega t + 5)^2 \omega^2 )</td>
</tr>
</tbody>
</table>

We can calculate the second derivative with respect to \( t \) of this expression by performing differentiation twice. We simplify the expression a
bit by performing the operation simplify->symbolic.

\( \frac{d}{d\omega} \sin(\omega t + 5)^2 = 2 \sin(\omega t + 5) \cos(\omega t + 5) \frac{d}{dt} \)

If we want the document to display the mathematical notation for the
derivative, we can select \( \frac{d}{dx} f \) from the expression palette and then
fill in the slots \( f \) and \( x \). We can then get Maple to calculate the derivative
by typing control-\textasciitilde.

**Symbolic differentiation and evaluation of derivatives, textually**

**Example**

\( expr := x^2 - 2x + 5 \)

Find the first derivative of the expression with respect to \( x \).

\( x^2 - 2x + 5 \) (19.91)

Find the first derivative of the expression with respect to \( t \).
Find the second derivative of the expression with respect to \( t \). The result is the same as if we had taken the derivative of 1.2.4.

\[
\text{diff}(\text{expr}, x)
\]

\[
2x - 2
\]

(19.92) \( \text{diff}'s \) second argument must be the variable of differentiation. Note that if the variable doesn't occur in the expression, the derivative (according to the mathematical definition) is zero. If you are surprised by getting a zero derivative, check that you are using the correct variable to differentiate with respect to.

\[
posExpr := \sin(\omega t + 5)^2
\]

\[
\sin(\omega t + 5)^2
\]

(19.93) \( \text{simplify} \) can reduce the size of the expression, although \( \text{factor} \) or \( \text{expand} \) sometimes work better. Sometimes additional trigonometric identities need to be applied, through \( \text{simplify}(..., \text{trig}) \).

\[
diff(posExpr, t)
\]

\[
2 \sin(\omega t + 5) \cos(\omega t + 5) \omega \]

(19.94)

\[
diff(posExpr, t, t)
\]

\[
2 \cos(\omega t + 5)^2 \omega^2 - 2 \sin(\omega t + 5)^2 \omega^2
\]

(19.95)

This is a way to compute \( \frac{d}{dx} x^2 - 2x + 5 \bigg|_{x = 3} \).

\[
diff(expr, t)
\]

0

(19.96)

\[
\text{simplify}((1.9.5))
\]

\[
2 \omega^2 (2 \cos(\omega t + 5)^2 - 1)
\]

(19.97)

This is a way to compute \( \frac{d^2}{dt^2} \sin(\omega t + 5)^2 \bigg|_{t = 47} \). The result of evaluation does not have to be a number even if a numeric value is being supplied for one of the variables in the expression being evaluated.

\[
eval(1.9.2, x = 3)
\]

4

(19.98)

\[
eval(1.9.7, t = 47.0)
\]

\[
2 \omega^2 (2 \cos(47.0 \omega + 5)^2 - 1)
\]

(19.99)

### Textual version of limits

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lim(1/(x-3)^2, x=3) )</td>
<td>The first argument is the expression that you wish to take the limit of. The second argument is an equation indicating the limit variable and the limiting value.</td>
</tr>
<tr>
<td>( \lim(\sin(x)/x, x=\text{infinity}) )</td>
<td>You can use &quot;infinity&quot; or &quot;-infinity&quot; as the way of specifying ( \infty ) or (-\infty) textually without use of the palette.</td>
</tr>
<tr>
<td>( \lim(1/t, t=0, \text{left}) )</td>
<td>The third optional third argument to ( \text{limit} ) can specify a one sided limit.</td>
</tr>
<tr>
<td>( \lim(1/t, t=0, \text{right}) )</td>
<td>This is the textual way of specifying ( \lim_{t \to 0^+} \frac{1}{t} ). There is no simple way of specifying this</td>
</tr>
<tr>
<td>( \lim(1/t, t=0) )</td>
<td>This is the textual way of specifying ( \lim_{t \to 0} \frac{1}{t} ).</td>
</tr>
</tbody>
</table>

### Minima and maxima

<table>
<thead>
<tr>
<th>Operation</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minima</td>
<td></td>
</tr>
<tr>
<td>Maxima</td>
<td></td>
</tr>
</tbody>
</table>
### Exact minimum or maximum value

\[ f := (x) \rightarrow 2 - 2 \cdot \sin(x) \text{ for } x \text{ in } -\frac{\pi}{4} \ldots \pi \]

**Solution**

\[ f := (x) \rightarrow 2 - 2 \cdot \sin(x) \]

\[ x \rightarrow 2 - 2 \sin(x) \quad (19.104) \]

**plot** \( f, -\frac{\pi}{4}, \pi \) It's usually a good idea to plot the objective function to get some sense of where the extrema are. Remember that the function to be investigated should be continuous if you expect good results from calculus techniques.

0

\[ \text{largestValue := maximize } \left( f(x), x = -\frac{\pi}{4}, \pi \right) \quad (19.105) \]

2 + \sqrt{2}

\[ \text{minResult := minimize } \left( f(x), x = -\frac{\pi}{4}, \pi, \text{location} \right) \quad (19.106) \]

### Exact minimum or maximum value location

0, \[ \left\{ \left\{ x = \frac{1}{2} \pi \right\}, 0 \right\} \]

\[ \text{maxResult := maximize } \left( f(x), x = -\frac{\pi}{4}, \pi, \text{location} \right) \quad (19.107) \]

2 + \sqrt{2}, \[ \left\{ \left\{ x = -\frac{1}{4} \pi \right\}, 2 + \sqrt{2} \right\} \]

\[ g := (t) \rightarrow \sin(2 \cdot t) \quad (19.108) \]

### Evaluation of a function at an extremum point

\[ t \rightarrow \sin(2 \cdot t) \quad \text{Evaluate } g \text{ at the location of the smallest value of } f. \]

\[ \text{minLocation := rhs(minResult[2][1][1][1])} \quad (19.109) \]

\[ \frac{1}{2} \pi \]

\[ g(\text{minLocation}) \quad (19.110) \]

0

\[ \text{expr := } x^2 + \sin(x) \quad (19.111) \]

### Evaluation of an expression at an extremum point

\[ x^2 + \sin(x) \]

\[ \text{minEvalpt := minResult[2][1][1]} \quad (19.112) \]

\[ \left\{ x = \frac{1}{2} \pi \right\} \]

\[ \text{eval(expr, minEvalpt)} \quad (19.113) \]
\[ \frac{1}{4} \pi^2 + 1 \]

with \textit{Optimization}:

\begin{align*}
\text{approxMinResult} & := \text{Minimize} \left( f(x), x = -\frac{\pi}{4}, \pi \right) \\
& \approx [0., [x = 1.57079632679489]] \\
\text{approxMaxResult} & := \text{Maximize} \left( f(x), x = -\frac{\pi}{4}, \pi \right) \\
& \approx [3.41421354582207, [x = -0.785398151694103]] \\
\end{align*}

\begin{align*}
g(\text{rhs(approxMaxResult[2][1]))} & \approx -1.000000000000000 \\
\text{eval(expr, approxMaxResult[2])} & \approx -0.902565162265192 \\
\end{align*}

Remember that numerical methods can be fooled into returning answers that are seriously incorrect. Verify your computed results.
20 More mathematical modeling: area-finding, piecewise expressions, and splines

20.1 Chapter Overview

We introduce three more computational features from Maple’s built-in library:

- Computing definite and indefinite integrals
- Piecewise expressions
- Curve fitting using splines

Many scientific or engineering problems benefit from the use of these mathematical concepts.

20.2 Recalling basic facts about integration

We review some of the basic facts and terminology about integration from your calculus courses.

**Definite** can be related to finding the area under a curve described by a function. The **definite integral** \( \int_a^b f(x) \, dx \) is notation for this. If an answer for a definite integration problem exists, then it is a number which could be described exactly or as an approximation.

**Indefinite integration** is sometimes called the problem of antiderifferentiation: given a function \( f \), find another function \( F \) whose derivative is \( f \):

\[
\frac{d}{dx} F(x) = f(x) .
\]

Another way of writing this relationship is:

\[
\int f(x) \, dx = F(x) .
\]

In most cases, there is no single function \( F(x) \) that solves the antiderivative problem for \( f(x) \). There is a whole family of functions that differ from each other by a constant. Sometimes the antiderivative \( F(x) \) is written with an extra symbolic constant:

\[
\int f(x) \, dx = F(x) + C ,
\]

particularly if antiderifferentiation is going to be applied as only one step of a problem-solving process.

Since the hard part of doing indefinite integration is finding \( F(x) \) without the constant and since the name of the constant can be freely chosen, systems that can compute antiderivatives often omit the constant in their answers, letting the user add in the constant of their choice later if they want to.

The answer \( F(x) \) is expected to be a formula involving expressions and functions that the user already knows about. \( F(x) \), when it exists, is referred to as a *closed form solution to the integration problem* or the *antiderivative of \( f(x) \)*. For calculus freshmen, these are expressions that involve numbers, polynomials, roots, rational functions, trig, logarithm and exponential functions. For more advanced mathematics students, it might include other functions: *erf* (error function), *Bessel*, \( \Gamma \), etc.

The **Fundamental Theorem of Calculus (FTC)** states that the definite integral can computed if the antiderivative is known:

\[
\int_a^b f(x) \, dx = F(b) - F(a) ,
\]

where \( F \) is any function such that

\[
\frac{d}{dx} F(x) = f(x) .
\]
Programs for doing indefinite integration/antidifferentiation have been around from the early '60s. They can be found in a number of symbolic systems: Maple, Mathematica, Macsyma (now found in SAGE), and MuPad (now part of Matlab). It is not an easy program to write. Even today in the era of "open source", the quality and power of such programs varies significantly.

The ideas for how to program an indefinite integrator comes from seminal work done by Robert Risch in the late 1960s, who was in turn continuing work done previously by other mathematicians such as Joseph Liouville (1809-1882). Earlier approaches in the early '60s treated symbolic anti-differentiation as a problem in artificial intelligence. Those AI programs (such as SAINT by James Slagle (1934-1994) and SIN by Joel Moses (1941-) tried to simulate the expertise of a college calculus student. However, they would sometimes give up on problems that could be solved through a clever trick of substitution or other reformulation. Risch's work used advanced mathematics to come up with programs that would guarantee that they could find a solution if there was one to be found, for certain classes of integrands. However, while programs using Risch's techniques are more powerful, they don't necessarily give the answer in simplest terms as a mathematics textbook would. That may require more symbolic manipulation, under human guidance.

20.3 Indefinite integration

Doing anti-differentiation with int

Table 20.1: Indefinite integration with int

<table>
<thead>
<tr>
<th>Indefinite integration with int</th>
</tr>
</thead>
<tbody>
<tr>
<td>The clickable interface can be used to input an indefinite integration problem for Maple to compute. Use the ( \int ) item in the Expression Palette, then edit the expression ( f ) and the variable of integration ( x ). This computes the antiderivative of the expression, with respect to the variable of integration.</td>
</tr>
</tbody>
</table>

The general form of the textual version of the integration command is

\[
\text{int( expression , var )}
\]

Table 20.2: Indefinite integration with int

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
</table>
| \[
\int \frac{1}{x} \, dx
\]
| \( \ln(x) \) | (20.1) |
| \[
\int \cos(x)^2 \, dx
\]
| \( \frac{1}{2} \cos(x) \sin(x) + \frac{1}{2} x \) | (20.2) |
| \[
\int \cos^2(x) \, dx
\]
| \( \frac{1}{2} \cos(x) \sin(x) + \frac{1}{2} x \) | (20.3) |
| \[
\text{int(sqrt(x), x)}
\]
| \( \frac{2}{3} x^{3/2} \) | (20.4) |

The clickable interface can be used to enter the integration problem. In the clickable interface there are two ways of entering what is " \( \cos(x) \) times \( \cos(x) \) ".

The textual form of integration works in the same way.
This uses the textual form of the command in a small script. The script computes the symbolic integral and stores it as the value of the variable i1. Then it takes the derivative of i1 and assigns that to d1.

\[
\text{expr := tan(t) + x^2;}
\]
\[
i1 := \text{int(expr, t);} \\
d1 := \text{diff(i1, t);} \\
difference := d1 - \text{expr;}
\]
\[
\text{printf("difference: \%a\n", difference);}
\]
\[
\text{printf("simplified difference: \%a\n",}
\text{simplify(difference));}
\]

This uses the textual form of the command in a small script. The script computes the symbolic integral and stores it as the value of the variable i1. Then it takes the derivative of i1 and assigns that to d1. According to the Fundamental Theorem of Calculus, d1 should be the same as expr. However, we have to get Maple to simplify the difference in order to have it actually be reduced to zero, using the built-in library procedure `simplify`.

\[
\text{difference: sin(t)/cos(t)-tan(t)}
\]
\[
\text{simplified difference: 0}
\]

### Understanding the answers from int

If Maple does not know how to find an answer it will return an expression that is the display version of the input expression. It does this instead of giving an error message or NULL so that a script or program using integration can continue even if the attempt to integrate fails. For example, even if you don't have the anti-derivative of the integrand, you can still try to do an approximate numerical definite integration. We discuss that in section 20.4.

Sometimes the answer given by the symbolic integrator will be given in terms of functions that beginning students haven't heard of. This can be confusing, but unfortunately most systems written for professionals to use do not have a "beginners mode" that avoids using advanced math. If you see a function you don't know of, you can get a brief explanation of it through on-line help.

Whether or not the answer computed for you is satisfactory depends on the situation. A strange-looking answer might be a sign that you may have mis-entered the problem ("garbage in, garbage out"). It might mean that there's a way to simplify the strange-looking answer into something you recognize. If the answer involves functions you haven't heard of, it may mean that it's time to learn more math to deal with the situation.

To summarize, the responsibilities that users of symbolic integration programs have are a) to be prepared to learn about more advanced functions if answer is of that form, b) understand when the computer system is saying that it can't find a closed form solution.

### Table 20.3: Integration answers using advanced functions

<table>
<thead>
<tr>
<th>( \int e^x , dx )</th>
<th>(- \frac{1}{2} \sqrt{\pi} \text{erf}(1x))</th>
</tr>
</thead>
</table>

You wouldn't see this integration problem in an elementary calculus textbook, because the answer involves the "error function". The term "error function" is not indication that there is a mistake, it's the official terminology for the function, in the same way that "sine", "cotangent" or "exponential" are names of functions. The name arose from the function's original use in probability and statistics. You can read more
Integration answers using advanced functions

Maple can't find the antiderivative because we haven't told it what the function f is. (20.8)

It can't find the symbolic integral for g either, for the same reason. We see that printf says that the value of answer2 is, in textual form, the same as the input was.

Checking that your integration answer is correct

Like everything else produced by a computer, it's still your responsibility to verify that the computed answer is correct. After all, even if you think it unlikely that there's a programming error in int's programming, you might have mis-entered the function you want to integrate. However, it's easy to use Maple to check integration answers it provides:

Is the derivative of your answer the same or equivalent to what you started with?

Because of the relationship between integration and differentiation, it is possible to check integration answers without having to do the integration by hand:

1. Use int to calculate the answer
2. Apply diff to the answer. The result of differentiation should be the function you were originally tried to integrate with int.
3. If the derivative does not look the same as the integrand, try simplify on the difference between the result of step 2 and the original expression. If you get zero, the two are equivalent.

Table 20.5: Checking answers

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ expr := 1 + x + \frac{\cos(x)}{2} ]</td>
<td>The derivative of the symbolic integral is the same as the original expression. This is obvious just by &quot;eyeballing&quot; it, but we can also confirm that the difference between expr and the derivative is 0.</td>
</tr>
<tr>
<td>[ answer := \text{int}(expr, x) ]</td>
<td>[ x + \frac{1}{2} x^2 + \frac{1}{2} \sin(x) ]</td>
</tr>
</tbody>
</table>
The derivative of the symbolic integral does not appear at first glance to be the same as the original expression. However, the difference between them does simplify to zero, so they are equivalent. Maple does not automatically employ trig identities such as $\sin^2(x) + \cos^2(x) = 1$ because it would often introduce tedious delay to try to apply all possible identities all the time.

Here is another example of a symbolic integration answer whose derivative does not appear to be the same as the starting expression. However, simplification of the difference between the two indicates that the two are equivalent because their difference simplifies to zero.

Sometimes the "simplify the difference to zero" trick is not needed to check work because it is obvious that the two are the same. Maple has other simplification operations such as `normal`, `factor`, `convert`, `combine`, and `expand` that will attempt to put results into alternative forms. See on-line help about `simplify` and the other operations for more information.
Troubleshooting int for indefinite integration problems

`int` is intended to work with expressions involving the variable of integration, not names of functions. If an expression does not seem to involve the variable of integration, it is treated as a symbolic constant, regardless of whether it is also the name of a function. This can lead to misleading "answers"

Table 20.6: Variables assigned expressions work with int, but not variables assigned function definitions

<table>
<thead>
<tr>
<th>Variables assigned expressions work with int, but not variables assigned function definitions</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>badResult := \int \cos dt</code></td>
</tr>
</tbody>
</table>
| \[
| \cos t \\
|\] (20.24)                                                                                   |
| Since the integrand is just \( \cos \), and does not involve the variable of integration, Maple treats it as a symbolic constant. Since for a name \( a \), \[
| \int a \, dt = at \\
|\] the result of this integral is the name "\( \cos \)", times \( t \). It sort of looks like \( \cos(t) \) but it isn't. |
| `eval(badResult, t = 0)`                                                                    |
| \[
| 0 \\
|\] (20.25)                                                                                   |
| One way to demonstrate that this answer isn't \( \cos(t) \) is to evaluate it at \( t=0 \). \( 0 \cdot \cos = 0 \). That certainly isn't the value of \( \cos(0) \). |
| `eval(badResult, t = .5)`                                                                   |
| \[
| 0.5 \cos \\
|\] (20.26)                                                                                   |
| Evaluating this expression for another value indicates that Maple's integration result is actually "\( t \cos \)"; the variable \( t \) times the name of a function. |
| `goodResult := \int \cos(t) \, dt`                                                            |
| \[
| \sin(t) \\
|\] (20.27)                                                                                   |
| Giving the symbolic integrator an expression involving the variable \( t \) gives the desired result. |
| `eval(goodResult, t = 0)`                                                                   |
| \[
| 0 \\
|\] (20.28)                                                                                   |
| `eval(goodResult, t = .5)`                                                                   |
| \[
| 0.4794255386 \\
|\] (20.29)                                                                                   |
| `g := x \rightarrow x^2`                                                                    |
| \[
| x \rightarrow x^2 \\
|\] (20.30)                                                                                   |
| Another mistaken attempt to integrate a function name instead of an expression involving the integration variable. Like the example above, it produces the answer "\( g \) times \( x \)". Note that the value of \( g \) is not the expression \( x^2 \), it's the function definition \( x \rightarrow x^2 \). |
| `int(g, x)`                                                                                 |
| \[
| g \, x \\
|\] (20.31)                                                                                   |
| `int(g(x), x)`                                                                              |
| \[
| \frac{1}{3} x^3 \\
|\] (20.32)                                                                                   |
| Evaluating \( g(x) \) does produce \( x^2 \), which `int` can integrate properly.            |
| `int(g(t), t)`                                                                              |
| \[
| \frac{1}{3} t^3 \\
|\] (20.33)                                                                                   |
| Evaluate \( g(t) \) produces an expression in \( t \), which `int` can integrate properly if it is given \( t \) as the variable of integration. |
| `int(g(y), x)`                                                                              |
| \[
| y^2 \, x \\
|\] (20.34)                                                                                   |
| This is the integral of \( y^2 \) with respect to \( x \), which is \( x \) times \( y^2 \). |
Variables assigned expressions work with \textit{int}, but not variables assigned function definitions

\[
\text{int}(\text{abs}(b), b)
\]

\[
\begin{cases}
  -\frac{1}{2} b^2 & b \leq 0 \\
  \frac{1}{2} b^2 & 0 < b
\end{cases}
\]  \hspace{1cm} (20.35)

Integrating the absolute value function returns a piecewise expression. We talk more about piecewise expressions in Section 20.6.

\[
\text{procAbs} := \text{proc}(x) \text{ if } x \geq 0 \text{ then return } x \text{ else return } -x \text{ end if};
\]

We write a Maple procedure to emulate the absolute value function...

\[
\text{proc}(x) \text{ if } 0 \leq x \text{ then return } x \text{ else return } -x \text{ end if}
\]  \hspace{1cm} (20.36)

We can't integrate this procedure because the if statement gives an error when it tries to decide whether \( x \) is greater than zero when there is no numerical value for \( x \).

\[
\text{int(procAbs}(b), b)
\]

Error, (in procAbs) cannot determine if this expression is true or false: 0 <= b

\[
\text{procNew} := \text{proc}(x) \text{ evalf(exp(x)*a, a = 2*x) end proc}
\]

Maple can integrate the results of procedures, as long as the procedure can execute with symbolic-valued parameters.

\[
\text{int(procNew}(y + 1), y)
\]

This computes the integral of \( \exp(y+1)^*(y+1) \). Note that we could use it to calculate the integral of \( \exp(z)^2*z, \exp(w)^2*w \), etc. in a similar fashion.

\[
2 \exp(z+1)
\]  \hspace{1cm} (20.38)

\begin{table}[h]
\centering
\begin{tabular}{|l|l|
\hline
20.4 Definite integration & Table 20.7: Definite integration with \textit{int} \\
\hline
\hline
\textbf{Definite integration with int} & \\
\hline
To use the clickable interface to calculate a definite integral, use the \( \int_{a}^{b} f \, dx \) item in the Expression Palette, then edit the expressions \( f, a, \) and \( b, \) and the variable of integration \( x \) to suit your problem. \\
If the answer to a definite integration problem returns as the symbolic answer, then doing evalf( ...) of it will produce an approximation to the area,. These approximation techniques do not require the use of symbolic integration to find the answer. Approximation techniques will be used automatically, without evalf, if the integrand or the limits of integration involve floating point numbers rather than exact fractions. \\
The general form of the textual version of the command is \\
\text{int( expression , var = a..b )} \\
\hline
\end{tabular}
\end{table}

Table 20.8: Definite Integration Examples
### Example


Calculate \( \int_{-10}^{5} 6 \, dx \), \( \int_{-10}^{\pi/3} \frac{\pi}{3} \sin(x) \, dx \), \( \int_{0}^{3} \sqrt{\frac{x}{2}} \, dx \), \( \int_{0}^{2} \sqrt{4 - x^2} \, dx \).

- Using the clickable interface.
  - \( \int_{-10}^{5} 6 \, dx = 90 \) (20.39)

- Using the document interface but with the textual form of the operation.
  - \( \int \left( \sin(x), x = -\frac{\pi}{3}, \frac{\pi}{3} \right) \) = 0 (20.40)

- Using the clickable interface again.
  - \( \int_{0}^{3} \sqrt{\frac{x}{2}} \, dx = \frac{5}{2} \) (20.41)

- Entry of the operation within a code edit region, followed by its execution with control (command)-E

<table>
<thead>
<tr>
<th>operation</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>result := int(sqrt(4-x^2),x=0..2);</code></td>
<td>( \pi ) (20.42)</td>
</tr>
</tbody>
</table>

#### Commentary

**Anton problem 32 (a).** Calculate \( \int_{-3}^{3} \left( x^2 - 1 - \frac{15}{x^2 + 1} \right) \, dx \). Answer should be 30arctan(3/9) + 28/3.

- `int(abs(x^2 - 1 - 15/(x^2 + 1)), x=-3..3)`
  - \( \frac{28}{3} - 30 \arctan(3) + 60 \arctan(2) \) (20.43)

- `simplify((1.45))`
  - \( 30 \arctan \left( \frac{13}{9} \right) + \frac{28}{3} \) (20.44)

**Anton problem 32 (b).** Calculate \( \int_{0}^{\sqrt{2}} \left| \frac{1}{\sqrt{1 - x^2}} - \sqrt{2} \right| \, dx \) using Theorem 6.5.5. You will find it necessary which states that for \( a < b < c \), \( \int_{a}^{c} f(x) \, dx = \int_{a}^{b} f(x) \, dx + \int_{b}^{c} f(x) \, dx \). Integration is a domain where even the best programs may not be able to find the answer. It usually doesn't hurt to see if Maple's software can find the answer by itself. But we see here that since the result is the same as the input, that Maple can't do the problem without help.
to choose a good splitting point -- try plotting the function and finding where it is zero.

\[
expr := \text{abs}\left(\frac{1}{\sqrt{1-x^2}} - \sqrt{2}\right)
\]

\begin{equation}
int(expr, x = 0..\sqrt{3}/2)
\end{equation}

\[
\int_{0}^{\frac{1}{2}\sqrt{3}} \left| \frac{1}{\sqrt{1-x^2}} - \sqrt{2} \right| \, dx
\]

\(20.45\)

\(20.46\)

Plotting the function suggests that there's quite a change occurring around 0.7. This point is probably the splitting point that the problem is suggesting we find. But what is it, exactly?

Giving solve the problem of where the function is zero gives us two solutions. One is negative and we don't care about it since our definite integral is looking at the function only between 0 and \(\frac{\sqrt{3}}{2}\).

\[
zeroes := \text{solve}\left(\{\text{expr = 0}\}, x\right)
\]

\[
\left\{ x = \frac{1}{2}\sqrt{2}, x = -\frac{1}{2}\sqrt{2} \right\}
\]

\(20.47\)

\[
splitPt := zeroes[2]
\]

\[
\left\{ x = -\frac{1}{2}\sqrt{2} \right\}
\]

\(20.48\)

\[
bpt := \text{eval}(x, 1.4.10)
\]

\[
-\frac{1}{2}\sqrt{2}
\]

\(20.49\)
We use the splitting idea to ask Maple to evaluate two definite integrals. It can do each piece.

We predict that within 20 years Maple or similar systems will be able to do this problem completely on their own without this kind of manual analysis and manipulation.

Maple can do improper definite integrals since it can do symbolic limits (e.g. the functionality available through `limit`).

You can input infinity by through the Common Symbols Palette that can be found on the left hand side of the Maple application window.

If the coefficients or limits of integration are floating point numbers, then Maple may use alternative techniques to approximate the integral. These approximations are often good and sometimes work when the exact techniques do not.

### Numerical approximation of definite integration

We are unable to calculate the exact solution to this definite integration problem.

```
evalf of the expression for the integral will invoke a numerical approximation technique. It says that the area is approximately 694.15.
```

```
The difference between this and the original input is that the upper limit of integration has changed from "3" to "3.0". Because this is a floating point number, Maple will use numerical approximation automatically.
```
20.5 Arc-length integration

Recall the following information about the arc length of parametric curves (Anton, ch 7.4, theorem 7.4.3).

If no segment of the curve represented by the parametric equations

\[ x = x(t), y = y(t), a \leq t \leq b \]

is traced more than once as \( t \) increases from \( a \) to \( b \), and if \( \frac{dx}{dt} \) and \( \frac{dy}{dt} \) are continuous functions for \( a \leq t \leq b \), then the arc length \( L \) of the curve is given by

\[
L = \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
\]

We can use this formula and Maple’s definite integration operation to calculate arc length easily.

Maple has an arclength operation built into its "Student1" package:

**Table 20.9: Calculating arclength**

| Student[Calculus1][Arclength](x expression, y expression, parameter = a..b) |
calculates the arclength of a curve described by the parametric expressions in the two coordinates. The package uses the definite integration feature of Maple. Therefore, it will use approximate methods if \( a, b \), or any number in the expressions is a floating point number.

\[ \text{Student}[\text{Calculus1}][\text{Arclength}](f(x), x = a..b); \]
calculates the arc of the curve given by the parameterization \( x=t, y=f(t) \)

Table 20.10: Solving a problem involving arclength

<table>
<thead>
<tr>
<th>Problem 11, section 7.4 Anton p. 489:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let the parameterization be ( x = \cos(2t), y = \sin(2t), 0 \leq t \leq \frac{\pi}{2} ). Find the arclength.</td>
</tr>
</tbody>
</table>

\[
\text{plot}\left(\left[\cos(2t), \sin(2t), t = 0..\frac{\pi}{2}\right], \text{scaling = constrained}\right)
\]

\[
\text{Student}[\text{Calculus1}][\text{ArcLength}](\left[\cos(2t), \sin(2t)\right], t = 0..\frac{\pi}{2}) = \pi
\]

This is a parametric plot (see Drawing (page 127)). We say scaling = constrained so that the x and y axes are given the same scaling.

\[
\text{Student}[\text{Calculus1}][\text{ArcLength}](\left[\cos(2t), \sin(2t)\right], t = 0..\pi/2.0) = 3.141592654
\]

The use of "2.0" makes Maple use approximation techniques in the arclength calculation.

\[
\text{npExpr := sqrt(1-x^2)}
\]

\[
\sqrt{1-x^2}
\]

Solving a problem involving arc length

```
plot(npExpr, x=-1..1, scaling = constrained)
```

This is an alternative formulation for the same curve, as the plot indicates.

We load in the `Student[Calculus1]` package using `with`. Then we can just use `ArcLength` instead of the full name of the library procedure. `with(Student[Calculus1])`

```
```

```
ArcLength(npExpr, x=-1..1)
```

\[ \pi \]
20.6 Piecewise expressions and piecewise functions

Mathematical situations sometimes need functions that have more than one part to their definition. Consider the following expression:

\[
a := \begin{cases} 
-x & x < 0 \\
 x & x \geq 0
\end{cases}
\]

This defines a piecewise expression. It means "-x if x is less than 0, x if x is \( \geq 0 \)."

Table 20.11: Piecewise expressions

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
</table>
| \[ a := \begin{cases} 
-x & x < 0 \\
 x & x \geq 0
\end{cases} \] | Define a piecewise expression. It means "-x if x is less than 0, x if x is \( \geq 0 \)." |

\[ \text{plot}(a, x = -5..5) \]

Plot the piecewise expression for x in the range -5..5.

\[ b := \text{piecewise}(x < 0, -x, x \geq 0, x) \]

\[
\begin{cases} 
-x & x < 0 \\
 x & 0 \leq x
\end{cases}
\] \hspace{1cm} (20.66)

b is assigned the same piecewise expression, using the textual way of entering the information.

\[ b\text{Minus1} := \text{simplify}(2 \cdot b - 1) \]

\[
\begin{cases} 
-1 - 2x & x < 0 \\
-1 + 2x & 0 \leq x
\end{cases}
\] \hspace{1cm} (20.67)

Maple can do limited algebra on piecewise expressions.
Since $x$ is greater than or equal to 0, the bottommost piece of the expression applies, and the result of evaluation is $-1 + 2 \cdot 3 = 5$.

We can provide more branches to a piecewise expression. In the Palette version, you can add an extra row by typing control-shift-R (on Macintosh, command-shift-R) in the bottom row. In the textual version, you can add extra items onto the sequence of inputs to the function.

Table 20.12: More extensive piecewise expressions with the clickable and textual interfaces

<table>
<thead>
<tr>
<th>Clickable interface</th>
<th>Commentary</th>
</tr>
</thead>
</table>
| $c := \begin{cases} 
1 & x < -1 \\
2 & 1 > x \geq -1 \\
\varepsilon & x \sim 1
\end{cases}$ | To get this we typed $c :=$, then selected the \(-x \ x < a\) item from the palette. We edited what was there. Then we typed control-shift-R and filled in the bottom row. After completing the piecewise expression, we right-clicked (control-clicked in Mac) the expression and selected the 2D plot item from the popup menu. |

<table>
<thead>
<tr>
<th>Textual interface</th>
<th>This is an expression with three pieces that fit together at -1, and 2. To get the (\leq), we typed the characters (\leq) consecutively. Maple automatically reset them as the mathematical symbol (\leq). This uses the plot facility using the textual interface. We can evaluate the expression at (x=1.5). Since the input is limited-precision and the expression can be completely evaluated using arithmetic, we get a limited-precision result.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$step2 := piecewise\left(x &lt; -1, 1, -1 \leq x &lt; 2, x^2, x \geq 2, x^3\right)$</td>
<td>This is an expression with three pieces that fit together at -1, and 2. To get the (\leq), we typed the characters (\leq) consecutively. Maple automatically reset them as the mathematical symbol (\leq).</td>
</tr>
</tbody>
</table>
| \begin{align*}
1 & \ x < -1 \\
2 & \ -1 \leq x \text{ and } x < 2 \\
3 & \ 2 \leq x
\end{align*} | (20.69) |
| \begin{align*}
5 & \ x = 3
\end{align*} | (20.68) |
It is possible to use `otherwise` as one of the clauses of a piecewise expression. Usually there are an even number of inputs to `piecewise`, but if there is an odd number, then the last input is the "otherwise" value.

Table 20.13: More extensive piecewise expressions with the clickable and textual interfaces

<table>
<thead>
<tr>
<th>Clickable interface</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>plot(step2, x = -2 .. 4)</code></td>
<td></td>
</tr>
<tr>
<td><code>eval(step2, x = 1.5)</code></td>
<td>2.25 (20.70)</td>
</tr>
</tbody>
</table>

To get this we typed `c :=`, then selected the \( x < a \) item from the palette. We edited what was there. Then we typed control-shift-R and filled in the bottom row. After completing the piecewise expression, we right-clicked (control-clicked in Mac) the expression and selected the 2D plot item from the popup menu.
We define an expression that is \( \sin(x) \) for \( x \) between \( 0 \) and \( 5 \), \( 0 \) between \( \pi \) and \( 5 \), and \(-2\) everywhere else. If the number of terms in the sequence of inputs to \texttt{piecewise} is odd, then the last term is the "otherwise" clause.

Since a piecewise expression is still an expression, we can use it as part of a function definition. Within a function definition, a piecewise expression can be used instead of an \texttt{if then ... end if}. 
Table 20.14: More piecewise function definitions and uses

<table>
<thead>
<tr>
<th>Clickable interface</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ h := x \rightarrow \begin{cases} x &amp; x &lt; 1.5 \ x \cdot \log(x) &amp; x \geq 1.5 \end{cases} ] (x \rightarrow \text{piecewise}(x &lt; 1.5, x, 1.5 \leq x, x \log(x))) (20.73)</td>
<td>To get this we selected the (f := a \rightarrow y), from the Expression Palette, and then filled in (f) for (f), and (x) for (a). Positioning the cursor at (y), we then selected the (-x \cdot x &lt; a \leq ) item from the palette. We edited what was there.</td>
</tr>
<tr>
<td>(h(1.5))</td>
<td>(0.6081976622) (20.74)</td>
</tr>
<tr>
<td>(\text{solve}(h(x) = 3.2))</td>
<td>(2.954165523) (20.75)</td>
</tr>
<tr>
<td>(\text{plot}(h(x), x = 0..4))</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Textual interface</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>(h2 := x \rightarrow \text{piecewise}(x &lt; 1.5, x, 1.5 \leq x, x \cdot \log(x))) (x \rightarrow \text{piecewise}(x &lt; 1.5, x, 1.5 \leq x, x \log(x))) (20.76)</td>
<td>Maple formatted the (\rightarrow) as a right-arrow after we entered it. It also formatted (a \leq) that we typed as (a \leq).</td>
</tr>
<tr>
<td>(\text{solve}(h2(x) = 3.3))</td>
<td>(3.001983443) (20.77)</td>
</tr>
<tr>
<td>(h(x) - h2(x))</td>
<td>(0) (20.78)</td>
</tr>
</tbody>
</table>
20.7 Calculations with piecewise expressions

Many scientific and engineering situations can be modeled with piecewise expressions. Maple can apply `solve`, `diff`, and `int` on piecewise expressions, This can often save you time over trying to perform the operations on the pieces separately.

Table 20.15: Calculus on piecewise expressions and functions

<table>
<thead>
<tr>
<th>Calculus on piecewise expressions and functions</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Example</strong></td>
<td><strong>Problem:</strong> An ion-powered space probe is moving through a gaseous nebula. For the first ten hours the ion engine is on, and, it travels at a rate of $0.05 \cdot r^2$ meters per second. Then the motor shuts off and the gas in the nebula gradually slows the space probe down. The velocity of the space probe can be described as:</td>
</tr>
</tbody>
</table>
| | \[
| \begin{align*}
| 0 & \quad t < 0 \\
| \frac{5}{100} \cdot t^2 & \quad 0 \leq t < 10 \\
| \frac{5}{100} \cdot 10^2 \cdot \exp(-t + 10) & \quad t \geq 10
| \end{align*}
| | What is the position of the space probe at $t=20$ hours? |
| **Solution** | We assign the piecewise expression to the variable `velocity` and plot it to better understand it. The plot reveals that there are not any discontinuities in velocity, although there is an abrupt transition at $t=10$. |
| | `velocity :=`
| | \[
| \begin{align*}
| 0 & \quad t < 0 \\
| \frac{5}{100} \cdot t^2 & \quad 0 \leq t < 10 \\
| \frac{5}{100} \cdot 10^2 \cdot \exp(-t + 10) & \quad t \geq 10
| \end{align*}
| | (20.79) |
| | `plot(velocity, t = 0 .. 20)` |
| | (20.80) |
Calculus on piecewise expressions and functions

Example

\[
\text{position} := \int(\text{velocity}, t)
\]

\[
\begin{cases}
0 & t \leq 0 \\
\frac{1}{60} t^3 & t \leq 10 \\
-5 e^{-t + 10} + \frac{65}{3} & 10 < t
\end{cases}
\]

(20.81)

Since position is the integral of velocity, we can get an expression for the position by symbolically integrating velocity with respect to time.

To find the position at \( t = 20 \), we evaluate the expression \( \text{position} \) at that value. By giving the floating point number 20.0 instead of the exact number 20, we ensure that the result is given as a floating point number.

\[
\text{eval}(\text{position}, t = 20.0)
\]

21.66643967

(20.82)

We should get the same result if we integrate velocity for \( t = 0 \) to \( t = 20 \).

Table 20.16: Piecewise integration

| Problem 4(c) Anton, problem 6.7.42. A sprinter in a 100 m race explodes out of the starting block with an acceleration of \( 4.0 \frac{m}{s^2} \), which she sustains for 2.0s. Her acceleration then drops to zero for the rest of the race. (a) What is her time for the race? Make a graph of her distance from the starting block versus time. |
| Solution |
| Create a piecewise expression modeling the acceleration. |
| \[
\text{aexpr} := \text{piecewise}(0 \leq t \text{ and } t \leq 2, 4, 0)
\]

| \[
\begin{cases}
4 & 0 \leq t \text{ and } t \leq 2 \\
0 & \text{otherwise}
\end{cases}
\]

(20.84)
Piecewise integration

We integrate the acceleration expression to get a velocity expression.

\[ vexpr := \text{int}(aeexpr, t) \]

\[
\begin{cases}
0 & t \leq 0 \\
4t & t \leq 2 \\
8 & 2 < t
\end{cases}
\]  

(20.85)

We integrate the velocity expression to get a position expression.

\[ dexpr := \text{int}(vexpr, t) \]

\[
\begin{cases}
0 & t \leq 0 \\
2t^2 & t \leq 2 \\
8t - 8 & 2 < t
\end{cases}
\]  

(20.86)

We use \texttt{fsolve} to find approximately the time when the runner reaches 100 meters.

\[ \text{finishTime} := \text{fsolve}(dexpr = 100, t) \]

13.50000000  

(20.87)

We plot distance versus time to confirm the answer from \texttt{fsolve}.

\[ \text{plot}(dexpr, t = 0..\text{finishTime}, labels = ["time (in seconds)", "position (in meters)"]) \]

Table 20.17: Piecewise integration, continued

<table>
<thead>
<tr>
<th>Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two racers are in the race. One accelerates at 2m/sec(^2) for 4 seconds, as described previously. The other accelerates at 2.1 m/sec(^2) for 3.8 seconds. Who gets to the finish line first?</td>
</tr>
</tbody>
</table>
Solution
Create a procedure that turns the previous script into something that can handle other accelerations. The procedure should take two parameters, the initial acceleration rate \( a_{\text{Init}} \), and the period of time \( t_{\text{Init}} \) that this acceleration lasts. After the initial acceleration, the racer is assumed to run at the final velocity attained for the rest of the race. We want the procedure to return the time to complete the race. It should also, as a side-effect, print out the plot but it should not return the plot structure as a result.

\[
\text{racer := proc(} \ a_{\text{Init}}, \ t_{\text{Init}}\text{)} \\
\text{local aexpr, vexpr, dexpr, finishTime, plotStructure;}
\]

#Define acceleration
\[\text{aexpr := piecewise(} 0 < t \text{ and } t < a_{\text{Init}}, \ t_{\text{Init}}, 0);\]
#Calculate velocity and distance travelled.
\[\text{vexpr := int(aexpr, t);}\]
\[\text{dexpr := int(vexpr, t);}\]
#Calculate approximate time to finish.
\[\text{finishTime := fsolve(dexpr = 100, t);}\]
#Print out position vs. time.
\[\text{print(plot(} dexpr, \ t = 0 .. \text{finishTime, labels = ["time (in seconds)", "position (in meters)"]});\]
#Return finishing time.
\[\text{return finishTime;}\]
end proc:

#Test the procedure on the original problem
r1 := racer(2.0, 4.0);

\[
\text{r1 := racer(} 2.0, \ 4.0);\]

\[
\text{r2 := racer(} 2.1, \ 3.8);\]

#While the if statement is not really worthwhile if we only wanted to compare two racers, we are considering
#the possibility that we might want to do this
#frequently and so write for easy reuse.

if r1 < r2 then print("Racer 1 won!");
elif r1 > r2 then print("Racer 2 won!");
else print("They tied!");
end if;
Because piecewise expressions are similar to if then .. end if, it is tempting to think that they are interchangable. One thing to keep in mind is that operations such as int evaluate all their arguments before they start to try to figure out their results. If one of the arguments involves a procedure, then the procedure needs to work with symbolic rather than numeric arguments.
The difference between piecewise and a procedure with if then else.

\[
\text{int}(\text{abs}(b), b)
\]

Error, (in \text{int}) integration range or variable must be specified in the second argument, got piecewise\((x < 0, -x, 0 \leq x, x)\)

Integrating the absolute value function returns a piecewise expression.

We write a Maple procedure to emulate the absolute value function...

\[
\text{procAbs}(b)
\]

We find that Maple can't integrate it because it tries to get a formula for \(\text{procAbs}(b)\) and then integrate that. But the procedure can't run with \(b\) only a symbol.

\[
\text{proc}(x)
\]

\[
\text{proc}(x) \text{ if } 0 \leq x \text{ then return } x \text{ else return } -x \text{ end if; end proc}
\]

\[
\text{int}(\text{procAbs}(b), b)
\]

Error, (in \text{procAbs}) cannot determine if this expression is true or false:

\[
0 \leq \text{piecewise}(x < 0, -x, 0 \leq x, x)
\]

20.8 Curve fitting with splines

In scientific and engineering work, we often start with numerical measurements of a phenomenon, but need ways of knowing something about the mathematical behavior of the phenomenon for values other than those measured -- to make predictions, or to do mathematical modeling operations such as equation-solving, integration, or optimization that require formulae. A typical way of handling this is to construct a model (a formula) from the data.

Linear least squares data fitting, described in ???, is a way of doing this. However, it can only find formulas that are straight lines.

While there are techniques for non-linear least squares for certain kinds of models-- linear, quadratic, exponential, logarithmic, etc -- they all require that the modeler know ahead of time that whether the relationship should be linear, quadratic, or whatever. Picking the wrong kind of relationship may produce a procedure that doesn't very closely model the data.

There is a different kind of formula-fitting called spline interpolation. The advantage of the data fitting using splines is that the modeler does not have to know ahead of time whether the relationship is linear, quadratic, etc. A spline interpolant builds a piecewise expression that is guaranteed to agree exactly at every data point, instead of hopefully being close by as with least squares fitting.

Spline interpolants differ in the kind of function used for each piece of the expression. A linear spline connects the data points together with straight lines. That is, it is a piecewise expression where a linear expression describes each piece. A piece corresponds to the interval between two consecutive data points.

A quadratic spline is a piecewise expression where each piece connects two consecutive data points together with a degree 2 polynomials. Similarly there are cubic splines, quartic splines, etc. Typically quadratic or cubic splines are used for data fitting since they produce smooth curves that can be computed quickly.

Table 20.18: Linear splines

We enter a collection of data values

\[
dataValues := [[1, 3], [1.5, 2.8], [2.1, 4.6], [3.0, 4.4]]
\]

\[
dataValues := \text{[[1, 3], [1.5, 2.8], [2.1, 4.6], [3.0, 4.4]]}
\]
... and plot them.

\[
dataPlot := \text{plot}(\text{dataValues}, \text{style} = \text{point}, \text{view} = [1.4, 0.5, \text{symbol} = \text{diamond}, \text{symbolsize} = 20, \text{color} = \text{"Purple"})
\]

\[
\text{dataPlot} := \text{PLOT}(...)
\]

\[
\text{print(dataPlot)}
\]

We load in the CurveFitting package, and construct a linear spline which we call \( s \). Without the with, we could just do CurveFitting[Spline].

\[
\text{with(CurveFitting)}
\]

\[
[\text{ArrayInterpolation, BSpline, BSplineCurve, Interactive, LeastSquares, PolynomialInterpolation, RationalInterpolation, Spline, ThieleInterpolation}]
\]

\[
s := \text{Spline(dataValues, t, degree} = 1)
\]

\[
s := \begin{cases} 
3.400000000 - 0.400000000 t & t < 1.5 \\
-1.700000000 + 3.000000000 t & t < 2.1 \\
5.066666667 - 0.222222222 & \text{otherwise}
\end{cases}
\]
We display a plot of the spline, with the data point plot. This shows that the spline $s$ does agree with all the data points.

\begin{equation}
\text{plots[display]}([\text{plot}(s, t = 0 \ldots 4), \text{dataPlot}])
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{spline_plot.png}
\caption{Plot of the spline and data points.}
\end{figure}

To get a prediction of what to expect for say $t=2.3$, we just evaluate $s$ at that point.

\begin{equation}
\text{eval}(s, t = 2.3)
\end{equation}

\begin{equation}
4.55555556
\end{equation}

(2.3, 4.556) is on the line connecting the data points (2.1, 4.6) and (3.0, 4.4).

Maple's standard math operations know how to work with the piecewise expression that the spline generator produces.

\begin{equation}
\text{Optimization[Maximize]}(s, t = 1 \ldots 3.1)
\end{equation}

\begin{equation}
[4.5999999843952, [t = 2.10000000873216]]
\end{equation}

\begin{equation}
\text{int}(s, t = 1 \ldots 3.1)
\end{equation}

\begin{equation}
8.158888889
\end{equation}

\texttt{fsolve} will find one solution, but if we want the solution to be inside the range of values that we measured, we should specify the interval

\begin{equation}
\text{fsolve}(s = 3.1)
\end{equation}

\begin{equation}
0.7500000000
\end{equation}

\begin{equation}
\text{fsolve}(s = 3.1, t = 1 \ldots 4)
\end{equation}

\begin{equation}
1.600000000
\end{equation}

One problem with linear splines is that they can have sharp corners, as demonstrated by the example just given. It is possible, with sufficient data points, to construct splines where the pieces are quadratic, cubic, or of even higher degree. This has the advantage that the bends in the spline will be smoother. How smooth depends on the degree. However, higher degree splines may do undesirable
things for values outside the ranges being measured. For that reason, it's better to use splines for problems where you need results for values inside the ranges being measured.

Table 20.19: Quadratic splines and splines of higher degrees

The alternative form of spline construct takes two lists, one for first coordinate, and one for the second.

\[ tValues := [1, 1.7, 2.1, 3.7, 4.1] \]

\[ mValues := [3, 2.7, 4.8, 4.4, 3.1] \]

We can plot them.

\[ dataPlot2 := plot(tValues, mValues, style = point, view = [1 .. 8, 0 .. 8], symbol = diamond, symbolsize = 20, color = "Purple") \]

\[ dataPlot2 := \text{PLOT}(...) \]

With load in the CurveFitting package, and construct a quadratic spline which we call \( s2 \). Without the with, we could just do CurveFitting[Spline].

\[ \text{with(CurveFitting)} \]

\[ [\text{ArrayInterpolation, BSpline, BSplineCurve, Interactive, LeastSquares, PolynomialInterpolation, RationalInterpolation, Spline, ThieleInterpolation}] \]
We display a plot of the spline, with the data point plot. This shows that the spline $s$ does agree with all the data points.

The formula does have the drawback that it starts increasing after $t = 4.1$, even though the data doesn’t. This is a property of higher degree polynomials. We might not want to use this model if we need to extrapolate very far beyond the values that were measured.

Higher degree spline interpolation produces piecewise expressions that all agree fairly closely for values within the $t$ values measured within the region defined by the data points. However, they may differ significantly outside the region of the data points.

To get a prediction of what to expect for say $t = 2.3$, we just evaluate $s$ at that point.

$$\text{eval}(s, t = 2.3)$$

$$4.55555556$$

The formula does have the drawback that it starts increasing after $t = 4.1$, even though the data doesn’t. This is a property of higher degree polynomials. We might not want to use this model if we need to extrapolate very far beyond the values that were measured.
Table 20.20: Indefinite integration with int

The clickable interface can be used to input an indefinite integration problem for Maple to compute. Use the $\int f \, dx$ item in the Expression Palette, then edit the expression $f$ and the variable of integration $x$ to suit your problem.

The general form of the textual version of the command is

$$\text{int}(\text{expression} , \text{var})$$

This computes the anti-derivative of the expression, with respect to the variable $\text{var}$. If the answer is just a formula involving expressions that the user already knows about (e.g. ones that involve numbers, polynomials, rational functions, trig, logarithm and exponential functions), when Maple can find one, is sometimes referred to as a *closed form solution to the integration problem*.

<table>
<thead>
<tr>
<th>Indefinite integration with int</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\int \frac{1}{x} , dx$ $\ln(x)$</td>
<td>The clickable interface can be used to enter the integration problem. Note that in the clickable interface there are two ways of entering what is &quot;$\cos(x)\cdot\cos(x)$&quot;.</td>
</tr>
<tr>
<td>$\int \cos(x)^2 , dx$ $\frac{1}{2} \cos(x) \sin(x) + \frac{1}{2} x$</td>
<td>(20.106)</td>
</tr>
<tr>
<td></td>
<td>(20.107)</td>
</tr>
</tbody>
</table>
\[ \int \cos^2(x) \, dx \quad \frac{1}{2} \cos(x) \sin(x) + \frac{1}{2} x \quad (20.108) \]

\[ \int (\sqrt{x}) \, dx \quad \frac{2}{3} x^{3/2} \quad (20.109) \]

\[ \int \left( \cos(\alpha) \sin\left( \frac{\alpha}{2} \right), \alpha \right) \quad - \frac{1}{3} \cos\left( \frac{3}{2} \alpha \right) + \cos\left( \frac{1}{2} \alpha \right) \quad (20.110) \]

\[
\begin{align*}
\text{expr} & := \tan(t) + x^2; \\
i1 & := \text{int}(\text{expr}, t); \\
d1 & := \text{diff}(i1, t); \\
difference & := d1 - \text{expr}; \\
\text{print}(&\ "\text{difference:}", \ difference); \\
\text{print}(&\ "\text{simplified difference:}", \ \\
&\ \text{simplify}(\text{difference}));
\end{align*}
\]

This is use of the textual form of the command in a small script. The script computes the symbolic integral and stores it in i1. Then it takes the derivative of i1 and assigns that to d1.

According to the Fundamental Theorem of Calculus, d1 should be the same as expr. However, we have to get Maple to simplify the difference in order to have it actually be reduced to zero.

Definite Integration examples

<table>
<thead>
<tr>
<th>Example</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculate [ \int_{-10}^{5} 6 , dr \int_{0}^{x} \frac{\pi}{3} \sin(x) , dx \int_{0}^{3} \left</td>
<td>x - 2 \right</td>
</tr>
<tr>
<td>Entry of the operation within a code edit region.</td>
<td></td>
</tr>
<tr>
<td>[ \int_{-10}^{5} 6 , dr ]</td>
<td>90 \quad (20.112)</td>
</tr>
</tbody>
</table>
\[ \int \sin(x), x = \frac{\pi}{3}, \frac{\pi}{3} \]  
\[ = 0 \]  
(20.113)

\[ \int_{0}^{3} |x - 2| \, dx \]  
\[ = \frac{5}{2} \]  
(20.114)

\[ \int (\sqrt{4 - x^2}), x = 0 .. 2 \); 
\[ = \pi \]  
(20.115)

**Anton problem 6.6.25.** Calculate 
\[ \int_{1}^{4} \left( \frac{3}{\sqrt{t}} - 5\sqrt{t} - t^{-\frac{3}{2}} \right) \, dt \]

\[ \int \left(3/\sqrt{t} - 5\sqrt{t} - t^{-3/2}\right), t = 1 .. 4 \]
\[ = \frac{55}{3} \]  
(20.116)

**Anton problem 32 (a).** Calculate 
\[ \int_{-3}^{3} \left( x^2 - 1 - \frac{15}{x^2 + 1} \right) \, dx \]  
Answer should be 
30*arctan(13/9) + 28/3.

\[ \int (|x^2 - 1 - 15/(x^2 + 1)|), x = -3 .. 3 \]
\[ = \frac{28}{3} - 30 \arctan(3) + 60 \arctan(2) \]  
(20.117)

\[ \text{simplify}((1.9.12)) \]
\[ = 30 \arctan \left( \frac{13}{9} \right) + \frac{28}{3} \]  
(20.118)

**Anton Example 4, ch. 8** Evaluate 
\[ \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx \]

\[ \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \, dx \]
\[ = \pi \]  
(20.119)

**Anton Example 6 (c) Ch 8.** Evaluate 
\[ \int_{0}^{\infty} \frac{1}{\sqrt{x(x+1)}} \, dx \]

\[ \text{answer} := \text{int}(1/(\sqrt{x}(x+1)), x = 0 .. \text{infinity}); \]
\[ \text{printf}("\text{sin evaluated at the answer is: } %a", \text{sin(answer)}); \]
$$\pi$$

sin evaluated at the answer is: 0

### Approximate definite integration examples

<table>
<thead>
<tr>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int \left( \frac{\exp(x^2)}{\log(x^2)} \right) , dx ) from 1 to 3</td>
<td>( \exp(\pi) )</td>
</tr>
</tbody>
</table>

Maple is unable to calculate the exact solution to this definite integration problem. evalf says that the area is approximately 694.15.

Note that the difference between this and the original input was that we have changed the upper limit of integration from "3" to "3.0". Because this is a floating point number, Maple will use evalf automatically.

This is the same integrand. Note that the lower limit of integration is where the integrand is infinite.

Not surprisingly, the approximation method doesn't give us a number for the answer. Since the approximation method doesn't guarantee that its answer is close to the actual answer, this result doesn't prove that the area under the curve is infinite. Further analysis by a mathematician might establish this conclusively.

There is a symbolic solution to the integral, but it's messy. An explanation of "RootOf" can be found in Example 17.1.3.2.

As with the integrand above, making the upper limit of integration into a floating point number invokes the approximation technique that evalf uses.

$$\int \left( \frac{1}{x^2 + 2x + 1} \right) \, dx$$ from 0 to 1

Calculating arclength

<table>
<thead>
<tr>
<th>Command</th>
<th>Example</th>
</tr>
</thead>
</table>
| Student[Calculus1][Arclength]([ x expression, y expression], parameter = a..b) | }
calculates the arclength of a curve described by the parametric expressions in the two coordinates. The package uses the definite integration feature of Maple. Therefore, it will use approximate methods if \( a, b, \) or any number in the expressions is a floating point number.

\[
\text{Student}[	ext{Calculus1}][\text{Arclength}](f(x), x = a..b); \text{ calculates the arc of the curve given by the parameterization } x=t, y=f(t)
\]

\[
\text{Student}[	ext{Calculus1}][\text{Arclength}](f(x), x = a..b, showfunction=true); \text{ calculates the arc of the curve given by the parameterization } x=t, y=f(t)
\]

\[
\text{Student}[	ext{Calculus1}][\text{Arclength}](\{x \text{ expression, y expression}\}, \text{parameter } = a..b, \text{output=plot, showfunction=true, showintegral=true, showintegralrand=true});
\]

### Piecewise expressions

A piecewise expression uses different expressions to describe the value for different ranges of the controlling variable. It can be entered either with the clickable interface or textually.

To add more branches to a piecewise expression. In the clickable version, you can add an extra row by typing control-shift-R (on Macintosh, command-shift-R) in the bottom row. In the textual version, you can add extra items onto the sequence of inputs to the function.

If the word otherwise is used for a range in the clickable interface, then the expression associated with that branch will be used for all other values of the controlling variable. The otherwise branch is specified in the textual version of piecewise as the odd last argument to piecewise expression.

A difference between piecewise expressions and \textbf{if then ... end if} is that Maple can do mathematical operations such as +, solve, diff, int, or plot on piecewise expressions, while it can't do so as easily with \textbf{if then ... end if}.

\[
h := x \rightarrow \begin{cases} 
  x & x < 1.5 \\
  x \cdot \log(x) & x \geq 1.5 
\end{cases}
\]

\[
x \rightarrow \text{piecewise}(x < 1.5, x, 1.5 \leq x, x \log(x))
\]  

\[
h(1.5)
\]

\[
0.6081976622
\]  

\[
solve(h(x) = 3.2)
\]

\[
2.954165523
\]  

\[
\text{step3 := piecewise}(\ -2 \cdot \pi \leq x \leq \pi, \sin(x), \pi < x < 5, 0, -2)
\]

\[
\begin{cases} 
  \sin(x) & -2 \pi \leq x \text{ and } x \leq \pi \\
  0 & \pi < x \text{ and } x < 5 \\
  -2 & \text{otherwise}
\end{cases}
\]

(20.128)
(20.129)
(20.130)
(20.131)
plot(step3, x=-10..10)
21 Calculations that result in true or false

21.1 Chapter Overview

true and false are the two Boolean (sometimes known as "truth") values. Variables can be assigned Boolean values just as they can be assigned numeric values, formulae with symbols in them, equations, ranges, lists, etc. Boolean expressions are any expression which can take on a true or false values. Boolean expressions are commonly found in if or while statement as the conditions controlling execution of the statement. Functions can be defined in Maple that return a Boolean value as their result. Such functions are known as Boolean functions. Boolean functions are often useful in larger programming projects to give a easy way to invoke code where complicated decision-making occurs.

21.2 Boolean values and Boolean expressions

In contrast to numeric values (integer, fractions, floating point), or algebraic (\(x^2 + 1\)), we have the truth values, true and false. These are sometimes referred to as Boolean values, in recognition of the pioneering work into mathematical logic by the British mathematician George Boole (1815-1864) (ref: "BOOLE, GEORGE." Encyclopedia of Computer Science. Hoboken: Wiley, 2003. Credo Reference. 23 Oct. 2006. Web. 11 May 2010. <http://www.credoreference.com.ezproxy2.library.drexel.edu/entry/enencyc/boole_george>). Like other types of values, they can be assigned as the value of a Maple variable through :=.

In chapter 14, Table 14.10, we introduced the relational operators <, >, <=, etc. that could be used to create an equality that could be used as the condition in an if statement or while loop. Such conditions are a type of Boolean expression -- an expression whose value is true or false. One can create more complicated Boolean expressions by the use of the Boolean operators and, or, and not. Using such operators, one can combine Boolean expressions created using inequalities.

Boolean expressions and assignment of Boolean values to variables are often used in while statements, to calculate the next value of the controlling condition, in the midst of the loop. They can also be used to make the controlling logic of if statements more easily readable.

<table>
<thead>
<tr>
<th>Table 21.1: Operations that combine Boolean expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>and</strong></td>
</tr>
<tr>
<td>(a \text{ and } b) will be true if both (a) and (b) are true conditions.</td>
</tr>
<tr>
<td><strong>or</strong></td>
</tr>
<tr>
<td>(a \text{ or } b) will be true if either (a) or (b) (or both) are true conditions.</td>
</tr>
<tr>
<td><strong>not</strong></td>
</tr>
<tr>
<td>(\text{not}(a)) will be true if the condition (a) is false, and vice versa.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 21.2: A simple example using Boolean values and Boolean expressions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>isRaining := true</strong></td>
</tr>
<tr>
<td>(\text{true})</td>
</tr>
<tr>
<td><strong>isPouring := false</strong></td>
</tr>
<tr>
<td>(\text{false})</td>
</tr>
</tbody>
</table>
| **if** \(\text{isRaining \text{ and isPouring}}\) \then \text{oldManIsSnoring := true;}
| **else** \text{oldManIsSnoring := false; end if,}                     |
| \(\text{false}\)                                                     |

We can imagine writing a script doing some data analysis where at some point we determine that it’s raining. A little later we decide that the data also say that it isn't pouring.

This is a way to write code to set up a variable whose Boolean value is determined by the English nursery rhyme (http://en.wikipedia.org/wiki/It%27s_Raining,_It%27s_Pouring). While the rhyme doesn't say what happens if it isn't raining and pouring, we decide that we will assign the variable to have the opposite truth value of what it has when it is raining and pouring.
The condition isRaining and isPouring is false because isRaining and isPouring are not both true. Thus, the variable oldManIsSnoring is assigned false.

| oldManIsSnoring | false | We confirm that this is what the if statement did. |

Table 21.3: An example using boolean values and expressions

| Velocities := [95, 47, 86, 35, 16, 98, 87, 87, 90] |
| [95, 47, 86, 35, 16, 98, 87, 87, 90] |

| Colors := ["red", "green", "blue", "blue", "black", "red", "green", "white", "red"] |
| ["red", "green", "blue", "blue", "black", "red", "green", "white", "red"] |

| Licenses := ["PA", "ON", "PA", "NY", "HI", "BC", "NJ", "IL", "WY"] |
| ["PA", "ON", "PA", "NY", "HI", "BC", "NJ", "IL", "WY"] |

| Provinces := ["PEI", "NB", "QU", "NFL", "ON", "MAN", "SA", "AB", "YU", "BC"] |
| ["PEI", "NB", "QU", "NFL", "ON", "MAN", "SA", "AB", "YU", "BC"] |

| member("ON", Provinces) |
| true |

| member("MA", Provinces) |
| false |

| count := 0; |
| 0 |

| for i from | to | nops(Velocities) do |
| condition1 := Velocities[i] > 65 and not Colors[i] = "red" ; |
| condition2 := Velocities[i] < 50 and member(Licenses[i], Provinces); |
| if condition1 or condition2 then count := count + 1; end if; |
| end do; |

| printf("Number of licenses meeting criterion: \%d\n", count); |

| Number of licenses meeting criterion: 4 |

We have recorded the speed of cars going by an observation point (in miles per hour), along with their colors and their license plates. We want to count the number of cars that are either a) going faster than 65 mph and not red, or b) going slower than 50 miles an hour and are from Canada (British Columbia and Ontario are from Canada).

The built-in member function, returns true or false depending on whether the first argument to it, is an element of the second argument.

The condition isRaining and isPouring is false because isRaining and isPouring are not both true. Thus, the variable oldManIsSnoring is assigned false.

We confirm that this is what the if statement did.

We have recorded the speed of cars going by an observation point (in miles per hour), along with their colors and their license plates. We want to count the number of cars that are either a) going faster than 65 mph and not red, or b) going slower than 50 miles an hour and are from Canada (British Columbia and Ontario are from Canada).

The built-in member function, returns true or false depending on whether the first argument to it, is an element of the second argument.

The condition isRaining and isPouring is false because isRaining and isPouring are not both true. Thus, the variable oldManIsSnoring is assigned false.

We confirm that this is what the if statement did.
21.3 Boolean functions

We have seen how to design a function that returns a numerical result, and a plot structure result. Occasionally there is need to design a function that returns a Boolean result (either true or false). Such functions are used in larger programming projects where the decision-making is sufficiently complicated that readability may be enhanced by breaking off the code that does the decision-making and giving it a name. This allows it to be conveniently accessed and referred to, as well as reused if it is needed in several places inside the project.

While it is not a requirement, Boolean functions are often given a name that starts with the prefix "is", e.g. isPrime, isOdd, isDivisibleBy, isTouching, etc. This naming convention makes it easier to tell at a glance that the function is going to return a Boolean result.

The built-in function is takes a relationship such as an inequality and decides whether it is true or false. This can be helpful in the construction of functions that want to return a Boolean result without having an if statement around.

<table>
<thead>
<tr>
<th>Table 21.4: Example of a user-defined Boolean function</th>
</tr>
</thead>
</table>

Example of a user-defined Boolean function

```
isWithin := proc(a,b, x0,y0, distance)
    is(evalf(sqrt((a-x0)^2 + (b-y0)^2)<distance));
end proc;

proc(a, b, x0, y0, distance)
    is(evalf(sqrt((a-x0)^2 + (b-y0)^2) < distance))
end proc
```

We have taken a observations of a moving object in a two dimensional coordinate system. We want to find those that are within 5 meters of the (3,5). We write a boolean function that does something more general -- it returns true when the point (a,b) is within distance of the point (x0,y0).

```
Obs := [ [ 1,1,2,1], [ 1,3,2,4], [ 1,5,2,7], [ 1,9,2,9], [ 2,1,3,0], [ 2,7,2,5], [ 2,8,2,4] ]

[[ 1,1,2,1], [ 1,3,2,4], [ 1,5,2,7], [ 1,9,2,9], [ 2,1,3,0], [ 2,7,2,5], [ 2,8,2,4] ]
```

(21.12)

```
for i from 1 to nops(Obs) do
    a := Obs[i][1];
    b := Obs[i][2];
    if isWithin(a, b, 3, 5) then print(Obs[i], "is within", 2.5, "meters of", 3, 5);
    end if;
end do;
```

```
1.1
2.1
1.3
2.4
1.5
2.7
1.9
2.9
```

(21.13)
Now we want to find which points are within 2.6 meters of (2.7, 4.3). We learn from the code we wrote previously and come up with another version of the Boolean function that makes it a little easier. This version won't require us to unpack the coordinate first. It will do so inside the function.

$$\text{isWithin2 := proc(LO, x0, y0, distance)}$$
$$\text{is(evalf(sqrt((LO[1]-x0)^2 + (LO[2]-y0)^2)<distance))}$$
$$\text{end proc;}$$

$$\text{proc(LO, x0, y0, distance)}$$
$$\text{is(evalf(sqrt((LO[1]-x0)^2 + (LO[2]-y0)^2)<distance))}$$
$$\text{end proc}$$

Now we want to find which points are within 2.6 meters of (2.7, 4.3).

Table 21.5: A procedure that uses a Boolean function

<table>
<thead>
<tr>
<th>A procedure that uses a Boolean function</th>
<th>We define a function procedure that returns a list of all the points in L that are within distance of the specified point pt. This allows us to change the distance and the reference point at will without having to do any further reprogramming. We just invoke findWithin with different parameters.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$$\text{findWithin := proc(L, pt, distance)}$$</td>
<td></td>
</tr>
<tr>
<td>$$\text{local ptTab, i;}$$</td>
<td></td>
</tr>
<tr>
<td>$$\text{for i from 1 to nops(L) do}$$</td>
<td></td>
</tr>
<tr>
<td>$$\text{if isWithin2(L[i], pt[1], pt[2], distance)}$$</td>
<td></td>
</tr>
<tr>
<td>$$\text{then ptTab[i] := L[i];}$$</td>
<td></td>
</tr>
<tr>
<td>$$\text{end if;}$$</td>
<td></td>
</tr>
<tr>
<td>$$\text{end do;}$$</td>
<td></td>
</tr>
<tr>
<td>$$\text{return convert(ptTab, list);}$$</td>
<td></td>
</tr>
<tr>
<td>$$\text{end proc;}$$</td>
<td></td>
</tr>
</tbody>
</table>
distance1 := 2.5:
pt1 := [3,5]:
printf("%a are within %.2f meters of %a
", findWithin(Obs, pt1,distance1), distance1, pt1);

distance2 := 2.6:
pt2 := [2.7,4.3]:
printf("%a are within %.2f meters of %a
", findWithin(Obs, pt2, distance2), distance2, pt2);

distance3 := 2.1:
pt3 := [2.7,4.0]:
printf("%a are within %.2f meters of %a
", findWithin(Obs, pt3, distance3), distance3, pt3);

\[[[1.9, 2.9], [2.1, 3.0], [2, 7, 2.5]] \text{ are within 2.50 meters of } [3, 5]\]
\n\[[[1.3, 2.4], [1.5, 2.7], [1.9, 2.9], [2.1, 3.0], [2.8, 2.4]] \text{ are within 2.60 meters of } [2.7, 4.3]\]
\n\[[[1.5, 2.7], [1.9, 2.9], [2.1, 3.0], [2.8, 2.4]] \text{ are within 2.10 meters of } [2.7, 4.0]\]

21.4 Troubleshooting Boolean calculations

The most common mistake in doing a Boolean calculation is to use a result which Maple cannot decide is true or false. This is typically due to forgetting to assign a variable an expected numerical value, and then using a Boolean expression involving an inequality with the variable.

For i from 1 to nops(Velocities) do
if Velocities[i] < minUS and not(member(Licenses[i], Provinces)) then
minUS := Velocities[i];
end if;
end do;
printf("Minimum US speed observed is: \%dn", minUS);

Error, cannot determine if this expression is true or false: 95 < minUS
Minimum US speed observed is:
Error, (in printf) integer expected for integer format

Using the velocity/color/license plate information of the previous examples, we attempt to find the slowest speed by a non-Canadian car by writing a loop that updates the variable minUS whenever it finds a smaller value. However, there's an error because we forgot to initialize minUS, so when i=1, the truth or falsity of the expression Velocities[i]<minUS can't be determined. The computer is trying to decide whether 95<minUS is true or false, but since minUS has no assigned value, this can't be determined.

The printf statement also gives an error because it tries to print out the integer value of minUS, which has not assigned any numerical value by the time the printf statement is executed.
We fix this by initialize minUS to a value that we know will be larger than the eventual outcome.

```plaintext
minUS := 200;
for i from 1 to nops(Velocities) do
    if Velocities[i] < minUS and
        not(member(Licenses[i], Provinces))
    then minUS := Velocities[i];
    end if;
end do;
printf("Minimum US speed observed is: %d\n", minUS);

Minimum US speed observed is: 16
```

We create a Boolean function which, when given appropriate parameters, is supposed to decide whether the velocity parameter is less than the current minimum, and if it belongs to a Canadian car, returns true.

```plaintext
isCanadianMin := proc(velocity, license, minCAN, provList)
    return velocity < minCan and member(license, provList);
end proc;

minCAN := 200;
for i to nops(Velocities) do
    if isCanadianMin(Velocities[i], Licenses[i], minCAN, Provinces)
    then minCAN := Velocities[i];
    end if;
end do;
printf("Minimum Canadian speed observed is: %d\n", minCAN);

Error, cannot determine if this expression is true or false: 47 < minCan
Minimum Canadian speed observed is: 200
```

We note that the cause is the same as the previous example: minCan doesn't have a value so the computer can't decide whether 47 < minCan has no value. We fix the function to use the name of the parameter, minCAN, and things now work properly.

```plaintext
isCanadianMin := proc(velocity, license, minCAN, provList)
    return velocity < minCAN and
        member(license, provList);
end proc;

minCAN := 200;
for i to nops(Velocities) do
    if isCanadianMin(Velocities[i], Licenses[i], minCAN, Provinces)
    then minCAN := Velocities[i];
    end if;
end do;
```
```markdown
printf("Minimum Canadian speed observed is: %d\n", minCAN);

proc(velocity, license, minCAN, provList)
    return velocity < minCAN and member(license, provList)
end proc

200
Minimum Canadian speed observed is: 47
```

## 21.5 Chapter summary

<table>
<thead>
<tr>
<th>Boolean term</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>value</td>
<td>There are only two Boolean values in Maple: true and false. Maple variables can be assigned Boolean values.</td>
</tr>
<tr>
<td>expression</td>
<td>An expression whose value is either true or false. Maple variables can be assigned the result of evaluating Boolean expressions.</td>
</tr>
<tr>
<td>function</td>
<td>A function whose return value is either true or false</td>
</tr>
<tr>
<td></td>
<td>#Returns a result that is true or false depending on whether the point (a,b) is within the specified distance from the point (x0, y0)</td>
</tr>
<tr>
<td></td>
<td>isWithin := proc(a,b, x0,y0, distance)</td>
</tr>
<tr>
<td></td>
<td>is(evalf(sqrt((a-x0)^2 + (b-y0)^2)&lt;distance));</td>
</tr>
<tr>
<td></td>
<td>end proc;</td>
</tr>
</tbody>
</table>

### Boolean operations

- **and**  
  \( a \text{ and } b \) will be true if both \( a \) and \( b \) are true conditions.

- **or**  
  \( a \text{ or } b \) will be true if either \( a \) or \( b \) (or both) are true conditions.

- **not**  
  \( \text{not}(a) \) will be true if the condition \( a \) is false, and vice versa.

### Built-in Boolean functions

<table>
<thead>
<tr>
<th>Built-in Boolean functions</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>member(y; [x,y,z])</td>
<td>The function <em>member</em> tests whether something is an element of a set, and returns true or false accordingly.</td>
</tr>
<tr>
<td>true</td>
<td>(21.22)</td>
</tr>
<tr>
<td>is(15 &lt; 10)</td>
<td>The function <em>is</em> inspects its argument and decides whether the supplied expression is true or false.</td>
</tr>
<tr>
<td>false</td>
<td>(21.23)</td>
</tr>
</tbody>
</table>
21 Calculations that result in true or false
22 Continuing an education in technical computing

22.1 Chapter Overview

In this course, we have explored the technical features of a particular system for technical computing, Maple, and solved some problems that use mathematical models to explore and predict, and to design. We have emphasized those aspects of modeling that lead to understanding: the use of formulas, and summarizing numerical results through the use of graphs and animations. We found that certain kinds of mathematical operations come up frequently in mathematical modeling, such as solving equations, finding formulas to do data fitting, finding extrema, creating an animation, or evaluating a formula.

21st century technical workers can expect that the computation system they will provide pre-packaged programming to handle many of the basic needs of technical computing. Small calculations can be done interactively, but scripting and programming will be need to be done in a textual programming language for longer, more easily reused computations. The extra work involved in scripting or procedure development pays off when the computation needs to be reused several times, or when the results needs to be carefully documented to justify the decision-making based on it.

In addition to mathematical problem solving and programming notation, you have also experienced an incremental approach to software development, where testing is performed after each small step. Testing that each piece or step works before you proceed to the next often leads to faster and surer completion of the software project.

22.2 What you have done in the course: technical features

Over 12 labs and 12 homework/quiz sessions we have covered the following technical topics in technical computation:

a) Interactive calculation -- arithmetic, arithmetic with multitudinous math functions (trig, log, exp, roots), arithmetic with varying precision, exact arithmetic.

b) Intermixing of documentation and calculation so that results and their explanations can be performed in the same document where they are presented. We have seen the extra features useful for technical work, such as palettes for two dimensional technical notation, technical symbols, etc.

c) "Clickable" menus, and palettes to make entry of mathematics and operations require less typing and be more accurate.

d) Higher-level operations: solve, and plot. Eventually we found problems that motivated use of more of the large repertoire technical systems have: least squares and spline data fitting, or minimize and maximize, for example.

e) Scripts -- creating sequences of calculations to tackle longer or multi-part problems, parameterized to facilitate reuse.

f) Structures for holding collections of data: lists, plots, and tables. Part of the work in understanding this is how to initialize the structure, how to insert data into the structure, and how to extract it. Some of the concepts have interesting ramifications, such creating and using lists of lists to record things such as data points.

Maple has many more data structures -- vectors, matrices, and arrays for example, that more advanced applications motivate the use of.

g) One-line functions using ->.

h) Plotting and animation -- showing results to facilitates quick comprehension. In addition to function plotting, we have plotted data, done parameterized plotting, done multiple multi-color plots, and created animations of mathematically-described phenomena.

i) Programming -- looping and conditional execution. Encapsulating scripts into procedures (functions). Learning how to specify scripts through textual means, which meant getting used to a particular style of notation with balancing parentheses, a particular order of information, exactly correct spelling.

j) An expanded repertoire of mathematical representations. In addition to numbers, we have learned how to compute with formulae, such as calculus operations for differentiation and integration. Piecewise expressions can make it easier to represent information.
found in "real world" situations such as motivated by physics word problems. Allowing results to contain arbitrary symbols allows some symbols in certain formulae to represent units of measurement such as meters/hour rather than numbers.

k) On-line help and documentation. Much subsequent learning comes from learning from such sources, rather than having detailed personal tutoring.

l) "Widgets" for allowing simple ways of running and rerunning scripts with different values of parameters. The paradigm is to get a "off-the-shelf widget" and connect it to a computational procedure you have through small (one line) amounts of code. If a procedure produces the right kind of result, it can be displayed in another widget.

We have learned how to do these things using Maple, which supports all of this. But another key effect of our work is the ability to discuss and plan the work by getting good enough to decouple the implementation from the planning. In other words, when tackling a problem, the solution can be developed and stated in ordinary language, "define this equation", "plot the right hand side for x between 0 and 10", "solve the equation for x", with the understanding that it will be straightforward to translate these intentions into code that the computer will perform.

If you do not practice regularly, the particular details of how to do things in Maple such as plotting may fade from your memory. The important thing to remember when you return to the computer in the future after this course is over is that it is reasonable to expect to do these sorts of things without heroic effort, and that there are usually many examples in on-line help that you can use as a basis for (re)-learning and imitation. The upside of "so simple that even a freshman can do it" is that, if you've done your job as a student well, it is routine to come back and learn how to do it again if you've forgotten, or have to do it in another language. This is hardly unique to programming, it's true for all the mathematics and science that you learn now as a freshman and will have need to use again in later years.

22.3 What you have done in this course: how to work with computation

Besides our tour of "features you should expect to have, and how to get them", we have toured a variety of technical problems and the ways that computation can be used to solve them or understand them. A key skill in all of this is to be able to discover and state mathematical relationships that help to describe real situations -- the often painfully acquired "skill with word problems". These relationships have been referred to as a mathematical model. Sometimes finding the model is a matter of reading about it in an article created by an expert. The skill then is being able to translate the relationship into the syntax of whatever computer system you are using to do your work with. Sometimes you have to determine it yourself.

Once you gotten the description of the model into the computer, you have to figure out which operations (e.g. "fit this data to a line", "find right hand side of", "differentiate", "solve", "plot", "minimize", etc.) you want to perform with the mathematical model in order to find out the desired information. Often times the desired information is a prediction, which happens through a simulation. Extensive understanding of a phenomenon that can be modeled mathematically often involves many calculations.

Often tables, plots, or animations are produced to present the computed information in a form that more easily explains the results.

Having a good troubleshooting and test regimen is an important part of any effort to develop code that's more than a line or two long. An important thing to remember about your work in the labs is that the process was incremental -- build a little, then test and repeat. Getting evidence as you go that each step you build is sound before you put in the next one usually provides a much faster route towards a completed project, compared to testing only once you have developed and entered all the code.
Mathematical computation has become an essential part of most technical work. Expecting that one course will teach you everything that you need to know to use computation professionally is like expecting that one course in mathematics or There are several different additional ideas that a full education in computational science and engineering should include:

1. More sophisticated models of physical situations often require the use of linear algebra, and differential equations. Any of the technical systems (e.g. Maple, Matlab, Mathematica) can handle these calculations, either symbolically/exactly or with floating point calculation. It only awaits your becoming familiar enough with the mathematical ideas and taking the time to learn how the computational machinery you have already seen -- scripts, procedures, visualization, etc. -- is used with them.

2. More sophisticated calculations often require ideas from "advanced programming": recursion, object-oriented programming, tree and network-graph data structures. The history of computational science and engineering indicates that while for many decades mainstream applications avoided such features because of the additional effort needed by programmers to understand them, eventually the need for handling more sophisticated or complex situations has led to their extensive adaptation.

3. Computational efficiency becomes an issue as you write more code and rely less on built-in functions to handle major portions of your computation. The study of algorithms -- the development of a step-by-step description for how to solve a problem which can be straightforwardly turned into code --gives you access to the accumulated wisdom and cleverness about how to solve problems faster or using less memory. For large problems, the difference between efficient and inefficient methods can be striking -- seconds instead of hours or even years.

4. With calculations involving floating point numbers, numerical accuracy can sometimes be an issue. Relying upon built-in techniques such as fsolve or Optimization/Minimize gives you access to the techniques that are usually accurate, but sometimes built-in code is insufficient to solve your problem, and you must write your own. Poor approximation techniques can mean that your computed results are so far from the exact answer that they are useless for design or prediction purposes. The study of numerical analysis will give you access to what is known about getting good approximations and avoiding bad approximations with floating point numbers.

5. Several different types of computing have arisen that we haven't covered but are of growing value in technical fields: artificial intelligence -- the ability to get programs to tackle problems that most people regard as requiring thinking to solve, and data mining -- extracting conclusions from monumentally large amounts of information.

6. Most languages also support some form of parallel computing -- running parts of a program on several computer processors at once in hopes to getting results faster. For example, Maple can use all the processors of a "multicore" computer such as is typical on many personal computers nowadays through a parallel version of map.

If you want to teach yourself about these things, how do you go about doing it? We have seen that most computer systems have ample amounts of documentation explaining their features. There are many books and web pages devoted to explaining these kinds of ideas. It takes a familiarity with fundamental concepts (which this course has exposed you to), time, and motivation. The latter is particularly important. We recommend having a good excuse for learning the new material -- a new problem that you want to solve, for example. Then it's a matter of reading, experimentation, and talking to knowledgeable and responsive consultants when you get really stuck.

Proficiency in programming, like almost every other skill needs practice and periodic reflection on your progress. Making it a point to solve small problems as you encounter them will prepare you to do all the things necessary to handle larger projects. If you wait until an enormous project comes along to revisit programming, then you will find that an enormous amount of time will be needed to get up to speed. Incremental acquisition of knowledge is a way of living, not a course to take and be done with. It's the only way to become proficient with modern technical computing systems, given the large amount of functionality built into them and the large and diverse nature of problems they are used to help solve.

As you get more experienced and ambitious, you should also become familiar with more of the principles of software engineering. As might be expected, once a programming project encounters success, it attracts many users who want to reuse the software. They put pressure on the developers to fix and extend the programming to do more. The techniques used to design and maintain a large system -- say one that has $10^5$ - $10^7$ lines of programming, are far different from the informal and intuitive style we have used in this course. A program built for personal use -- taking a few hours to build -- is not at all like a large software systems
that takes dozens of programmers a few years to get working. Assuming that the "personal project" approach to software construction works on a large system makes about as much sense as assuming that the experience of building a backyard garden shed will be sufficient to let you build a 40 story skyscraper. You need organizational principles, mathematical acuity, a process for software development, and software tools to do extensive testing and recordkeeping, to get a large program to work and keep it functioning as needs and staffing changes. We've shown you a few software engineering ideas in this course -- frequent testing and incremental development, and the use of variable names that are words rather than single mathematical symbols. The field of software engineering can provide you with the best practices and formal analysis taken from the construction of large software systems over the past half century.

22.5 Conclusion

One course in computation is a start on the road to mastery, but it does not make you a master. This course has given you experience with the syntactic and semantic concepts that are found in most programming languages. Learning the second or further programming languages builds incrementally on what you have learned from working with the first language. Problem solving with a different language still requires first determining what you want to do, which is best expressed in your own language, not the computer's. The process of software development -- incremental development, with adequate description and plenty of incremental testing -- is the same regardless of which language you use.

The foundation has been laid for you to continue grow in your mastery of technical computing. You should be able to extend your proficiency to handle more sophisticated mathematics and mathematical models, using the syntax of whatever computer language you find the most expedient to use.
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