I. BACKGROUND

Archimedes’ Law of the Lever and Law of Buoyancy are two of the most fundamental laws of nature. In two of his works (On the Equilibrium of Planes I & II and On Floating Bodies I & II), Archimedes formulated these laws, deduced them from more fundamental principles, and applied them to determining the centers of gravity of various rigid bodies and determining the stability configuration of assorted floating bodies. These works of Archimedes are regarded as the foundations of the fields of static mechanics and hydrostatics.

In discussing Archimedes’ results on floating bodies, we will use certain modern language; for example, we refer to uniform and central gravitational fields and describe a body as having a weight vector pointing toward the center of Earth, although Archimedes and his contemporaries did not use or think in these terms. The use of such modern terminology, however, is for the convenience of the modern reader and does not invalidate the points we are trying to make.

Also, although Archimedes and his colleagues were well aware that Earth was spherical and that objects fall toward its center, in most of his works, he tacitly assumed that Earth was flat. That is, he assumed that the forces of gravity on separate pieces of a rigid body were parallel to each other and consequently that the magnitude of the resultant force was the sum of the individual magnitudes. He also assumed that the magnitude of the gravitational force on a body—its weight—did not change with the position or orientation of the body. As we would say today, Archimedes assumed that the rigid bodies he was examining were in a uniform gravitational field.

The one instance in his extant works where Archimedes did not assume a uniform gravitational field is in Propositions 8 and 9 at the end of his On Floating Bodies I (Ref. 5, pp. 261–262). These two propositions state that a solid body in the shape of a spherical cap will float stably on the surface of a spherical Earth with its base horizontal, either when the base is completely above the liquid level (Proposition 8) or below the liquid level (Proposition 9). Figure 1 shows photographs of such a body floating in water in these two horizontal-base positions.

To be precise, by a spherical cap, we mean either piece of a solid sphere (or ball) cut off by a plane. A spherical cap is also called a spherical dome, spherical segment, spherical section, or truncated sphere, although these usages are not consistent among authors. For brevity, we shall call a spherical cap simply a cap.

We shall call the center of the sphere from which the cap is formed the cap’s sphere-center. If the cutting plane passes through the sphere-center, either cap cut off is called a hemisphere. The flat portion of the surface of a cap is called its base and the line through the sphere-center and perpendicular to its base is called its axis.

In his Propositions 8 and 9, Archimedes assumed that Earth was a sphere and that a body of liquid on it would assume a spherical shape, in accordance with his Proposition 2, On Floating Bodies I (cf., Ref. 5, p. 254). For a particular cap, the vertical direction was determined by a line through its sphere-center and the center of Earth (the geocenter). He regarded a cap to be floating horizontally when its base was perpendicular to the vertical line.

Archimedes assumed throughout Book I that a particle is attracted toward the center of a spherical Earth and that the
magnitude of the force does not vary with the position of the particle or its distance from the geocenter. We make the same assumption, referring to this situation as Archimedes’ universe, as opposed to Newton’s universe in which the gravitational force varies with the inverse square of the distance of the particle from the geocenter. However, all of the results in this paper are qualitatively true for both universes. (See Ref. 11 for a derivation of Archimedes’ principle in a non-uniform gravitational field.)

Archimedes made a fundamental mistake in his proof of Propositions 8 and 9, a mistake that is quite often made even today. He assumed that the line of action of the gravitational force on a body in a spherical (or central) gravitational field goes through the body’s center of mass. This has been known to not be true since the time of Newton and a simple example is given in Appendix A. In Sec. II, it is shown how this mistake invalidates Archimedes’ proof.

In his works, Archimedes uses the Greek phrase κάντρον τοῦ βάρεος (kentron tou bareos), which is usually translated as “center of gravity” and occasionally as “center of mass.” However, in this paper, we shall translate it as “center of weight.” The word βάρος (baros) has the meaning of weight, bulk, mass, heaviness, burden, etc., and any of the translations is valid depending on context. In modern physics, only “center of gravity” and “center of mass” are used. The translation “center of gravity” should only be used in the context of a uniform gravitational field. In a uniform gravitational field, the center of gravity and the center of mass of a body reduce to the same thing, namely, a fixed point in the body through which the line of action of the gravitational force on the body passes, regardless of the position or orientation of the body. In a non-uniform gravitational field, the center of mass remains the same point (its definition does not depend on the existence of a gravitational field), but the concept of a center of gravity no longer exists, as is discussed in detail later on. In a central gravitational field, the line of action of the gravitational force on a body depends on the position and orientation of the body and generally these lines of action do not all pass through a fixed point in the body. In fact, a gravitational field is uniform if and only if for any rigid body there is a fixed point within it through which all lines of action of the gravitational force pass, regardless of the position or orientation of the body. That fixed point is then also the center of mass of the body.

Archimedes spoke only of a “center of weight” of a body and the properties he assigned to it are those possessed by its center of mass. When a body he was investigating was immersed in a uniform gravitational field, his center of weight was thus the same as both the center of gravity and the center of mass of the body. But for the one time he considers a body immersed in a central gravitational field (Propositions 8 and 9), his “center of weight” must be given the modern translation of “center of mass” since then the body no longer has a “center of gravity.”

Before examining Archimedes’ proof of Proposition 8, it should be emphasized that the purpose of this paper is not to criticize Archimedes or argue that he should have known the difference between the center of mass and the center of gravity of a rigid body. This would be analogous to criticizing Newton for not knowing or anticipating Einstein’s theory of relativity. The mechanics of Archimedes and his contemporaries was eventually superseded by the mechanics of Newton, and the mechanics of Newton was eventually superseded by the mechanics of Einstein. Just as the mechanics of Newton works brilliantly for objects travelling at speeds much less than the speed of light, the mechanics of Archimedes (i.e., that the line of the gravitational force always passes through the center of mass) works brilliantly for objects in a non-uniform gravitational field if the variation in the field is small with respect to the size of the object. Archimedes’ experience with objects moving in Earth’s central gravitational field was only with objects small with respect to Earth and such objects can be treated as if they were in a uniform gravitational field. It would have been just as difficult for Archimedes in his day to develop the mechanics of Newton as it was for Newton in his day to develop the mechanics of Einstein. Nevertheless, it is important for the modern reader to understand the differences among the laws of mechanics developed by these three individuals who lived centuries apart.

II. ARCHIMEDES’ PROOF OF PROPOSITION 8

Part I of supplementary material12 contains an English translation of Proposition 8 and its proof from the Greek text as edited by Heiberg13 from the Archimedes palimpsest.14–16 Figure 2 displays digital scans of three diagrams illustrating Archimedes’ proof, from a 1565 edition of Federico Commandino.17 Commandino illustrated three cases: when the cap is (a) more than a hemisphere; (b) equal to a hemisphere; and (c) less than a hemisphere.

Archimedes gave a detailed proof only for case (a) and in his last line states that cases (b) and (c) can be proved similarly. Heiberg did not include diagrams for the cases when the base is above the fluid level, but Fig. 3(a) is our diagram of case (a) based on Heiberg’s Greek text13 and Commandino’s diagram.

Figure 3(a) is a cross-section through the geocenter $A$ of Earth $AB\Gamma\Delta$ and the sphere-center $K$ that is perpendicular to the base $E\Theta H$ of the cap $EZH\Theta$. The line $\Theta Z$ is the axis of the cap and the line $KA$ defines the vertical direction. The axis $\Theta Z$ is shown tilted from the vertical direction with the base completely above the liquid level. In his proof, Archimedes attempts to prove that the forces acting on the cap will cause it to rotate counterclockwise so that its axis becomes aligned with the vertical (so that its base ends up horizontal). Because of the symmetry of Earth and the rigid body about the cross-section, all forces acting on the body will lie in the cross-section.

Archimedes first locates the center of mass (his “center of weight”) $P$ of the submerged portion of the cap, which is also the center of mass of the displaced liquid since both are of uniform density and occupy the same space. By symmetry, $P$ must lie on the vertical line $KA$ because the displaced liquid has rotational symmetry about this line. It must also lie within the submerged portion of the cap because that portion is a convex body (see Sec. II of the supplementary material12). It also follows from symmetry that the gravitational force acting on the submerged portion of the cap lies on the vertical line pointing toward the geocenter and the buoyancy force also lies on this vertical line, but pointing away from the geocenter.

In modern force-diagram terms, such as shown in Fig. 3(b), Archimedes constructs three forces acting on the cap: the vertical upward buoyancy force $B$, the vertical downward gravitational force $H$ of the submerged (dark) portion, and the gravitational force $E$ on the unsubmerged (light) portion. Notice that the magnitude of $B$ (the weight of the displaced
liquid) must be greater than the magnitude of \( H \) (the weight of the submerged portion of the cap), since the density of the cap is less than the density of the liquid.

Next, Archimedes determines the center of mass \( \Sigma \) of the unsubmerged portion of the cap by use of his Proposition 8 of *On the Equilibrium of Planes* I (Sec. III of the supplementary material\(^{15}\)). He correctly shows that \( \Sigma \) must lie to the left of the vertical \( K \Lambda \). His next step, however, is incorrect: he states that the gravitational attraction of the unsubmerged portion passes through \( \Sigma \). That is, the line of action of the force \( E \) passes through the center of mass of the unsubmerged portion of the cap. As previously discussed, this is generally not true in a central gravitational field (see Appendix A for further details). In fact, the line of the gravitational force actually passes to the right of \( \Sigma \) as we explain more fully later. As such, there arises the possibility that the line of action is along the vertical line \( \Theta \Lambda \). If that is the case, all three forces are along the vertical line and the cap is in equilibrium in a tilted position.

Archimedes finishes his proof with statements about the force \( E \) directing the left part of the cap downwards, while the force \( B \) directs the right part of the cap upwards, thus turning the cap to the horizontal-base orientation. Although not invalidating his proof, this kind of qualitative, fuzzy argument (although pioneering in Archimedes’ day) became obsolete once Newton gave the definitive formulation of the translational and rotational accelerations of a rigid body under the action of external forces in his *Principia Mathematica*.\(^{18}\)

### III. A CORRECT PROOF OF PROPOSITION 8

We now give a correct proof of the first part of Proposition 8, showing that the only equilibrium position of the cap when its base is above the liquid is when the base is horizontal. It is then shown that the horizontal-base equilibrium position is stable. The proof is valid for any cap with a uniform density distribution that is less than the density of the liquid. In fact, it is also valid if the density distribution of the cap is spherically symmetric about its sphere-center.

The gravitational field is assumed to be spherically symmetric and pointing toward the geocenter, but it may vary in magnitude with distance in any manner. In particular, the magnitude may be independent of the distance of the body from the geocenter, which is undoubtedly what Archimedes assumed.

Archimedes made no specific mention as to how the cap is released in the liquid. We shall make the assumption that the body is released from rest when the vertical component of the weight vector (i.e., the component in the direction of the line through the geocenter and sphere-center) is equal to the magnitude of the buoyancy force. This will prevent any immediate vertical acceleration of the cap’s center of mass upon its release.

We consider the three possible cases shown in Fig. 4. After tilting the cap and releasing it with its base above the liquid, the cross-section that passes through both the geocenter \( \Lambda \) and the axis of symmetry of the cap is examined. The line through these two centers constitutes the vertical of our discussion, and the line perpendicular to it constitutes the horizontal. The cap is tilted through an angle \( \theta \) from the horizontal so that the base remains completely above the liquid level.

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**Fig. 2.** Commandino’s diagrams illustrating Proposition 8 of Archimedes’ *On Floating Bodies* I (Ref. 17).

**Fig. 3.** (a) Diagram of Proposition 8 based on Heiberg text (Ref. 13). (b) Modern force diagram.
We next cut the cap by a plane perpendicular to the cross-section through the point $M$ where the vertical line cuts the base (or its extension, cf., Sec. IV of the supplementary material\textsuperscript{12}) and making the same angle $\theta$ with the horizontal line as the tilt angle. This plane decomposes the cap into the two shaded parts: a light part and a dark part. The light part lies to the left of the vertical line and the dark part is symmetric (its density distribution is symmetric) with respect to a plane passing through the vertical line and perpendicular to the plane of the figure.

By symmetry, the buoyancy force $B$ caused by the displaced liquid is along the vertical line in the upward direction. Similarly, the gravitational force $G$ on the dark shaded part of the cap will be along vertical line, but in a downward direction, because of its symmetric shape and symmetric density distribution.

The other part of the cap (the light shaded region) lies entirely to the left of the vertical line and so the gravitational force $F$ acting on it lies along a line through the geocenter $K$ to the left of the vertical line (cf., Postulate 7 of On the Equilibrium of Planes I in Sec. II of the supplementary material\textsuperscript{12}). Most importantly, as long as $\theta$ is not zero, the three forces acting on the cap cannot have a zero resultant, since two of them, $B$ and $G$, are vertical while the other, $F$, has a nonzero horizontal component. This means that the cap cannot be in equilibrium with its base above the liquid unless $\theta$ is zero.

In addition to being correct, this part of our proof is simpler than Archimedes’ proof. It does not require any concepts that Archimedes was not familiar with. In fact, it doesn’t even require his Proposition 8 of On the Equilibrium of Planes I (Sec. III of the supplementary material\textsuperscript{12}).

We next show that this horizontal-base equilibrium position when the base is above the liquid is stable. Such stability considerations are now commonly analyzed using energy arguments, but in keeping with the spirit of Archimedes’ proof, we shall show that tilting the cap with its base above the liquid level will cause the resulting forces and torques on it to return it to the equilibrium position. But we must draw upon results from Newton’s day to show this since Archimedes’ static mechanics was inadequate to handle forces that were not parallel. More specifically, Archimedes did not know how to find the resultant of several two-dimensional forces acting on a rigid body or know how the motion of the body is determined by the effect of the resultant on the center of mass.

The three forces acting on the cap in all panels of Fig. 4 all pass through the geocenter $\Lambda$ and consequently their resultant, the vector $H$ in Fig. 5, will also pass through that point. Because we assumed that the vertical components of the three forces add up to zero, $H$ can only have a horizontal component, and because only the force $F$ has a horizontal component, its horizontal component is $H$ and it points to the right. Also, because the horizontal line of action of $H$ lies below the center of mass $\Xi$, this force will produce a torque about it in the counterclockwise direction of magnitude $Hd$, where $d$ is the vertical distance from the geocenter $\Lambda$ to $\Xi$ and $H$ is the magnitude of the vector $H$.

From classical mechanics,\textsuperscript{19} under the action of the resultant force $H$, the cap’s center of mass $\Xi$ will have a horizontal acceleration to the right and the cap will have a rotational acceleration counterclockwise about $\Xi$. Each of the potential motions caused by these accelerations will separately cause a decrease in the angle $\theta$ in Fig. 5. Specifically, a horizontal movement of $\Xi$ to the right without any rotation will cause $\theta$ to decrease, and a counterclockwise rotation of the cap about $\Xi$, keeping $\Xi$ stationary, will also cause the angle $\theta$ to decrease. Thus, the angle $\theta$ will always decrease to zero and
so the body will come to rest (assuming some friction losses) with the base horizontal.

IV. COUNTEREXAMPLE TO ARCHIMEDES’ PROOF OF PROPOSITION 8

Mathematicians tend to critically examine a proof of a theorem to see what hypotheses were actually used, rather than what hypotheses were stated or assumed in the statement of the theorem. If there are hypotheses used that were not stated or assumed, then they must be added to the statement of the theorem to make it correct. But if a proof used fewer hypotheses than were stated or assumed, then the statement of the theorem can be strengthened to yield a more general or stronger result.

The latter is the case with Archimedes’ proof. Although he clearly assumed that the cap under consideration had uniform density, he did not need this assumption in his proof. His proof would still be valid (except for the location of the line of action of the gravitational force) if (1) the density of the cap was cylindrically symmetric about its axis, (2) the density of the submerged portion of the cap was uniform, and (3) the center of mass of the cap was below its sphere-center. Figure 6 shows one particular example of such a cap. This hemispherical cap has uniform density except for an embedded ring about its axis parallel to its base, which has a higher uniform density.

Our correct proof of Proposition 8 is not valid for the type of cap shown in Fig. 6, but Archimedes’ incorrect proof is. This leaves open the question of whether Proposition 8 is true for such a cap. The answer is no, and the mathematical details of a counterexample of a cap of the form shown in Fig. 6 are given in Appendix B.

Notice that our counterexample in Appendix B is to Archimedes’ proof of Proposition 8, not to his statement of Proposition 8. His statement of Proposition 8 is correct in spite of his proof, not because of it. If Archimedes’ proof were correct, it would follow that a cap as shown in Fig. 6 always floats stably with its base horizontal. The counterexample is thus an indirect way of showing that Archimedes’ proof cannot be correct.

V. A FLOATING CAP ON A FLAT EARTH

In 1650, Huygens discussed Archimedes’ work on floating caps and also considered other floating bodies (paraboloids, cones, cylinders, and planks). But he discussed them only for a flat Earth, thus avoiding the difficulties described in this paper. Similarly, in 1999, Stein discussed Archimedes’ proof of the stability of a floating cap, but only for the case of a flat Earth.

In fact, Archimedes’ proof of Proposition 8 is correct for a floating cap in a uniform gravitational field since then the centers of gravity and mass of the cap coincide. Indeed, his assumption that the gravitational force goes through the center of mass of the cap is equivalent to the assumption that the gravitational field is uniform, even though his figures [see Figs. 2 and 3(a)] illustrate a central field. Figure 7 illustrates this situation.

Figure 7(a) is analogous to Fig. 3(b). The tilted cap is immersed in the liquid so that there is no net force in the vertical direction, and so the magnitudes of the weight vectors E and H of the unsubmerged and submerged parts of the cap add up to the magnitude of the buoyancy vector B. Although the resultant of the forces is zero, there is a net counterclockwise torque about the center of mass of the cap. This torque rotates the cap about its center of mass so that it will eventually come to the horizontal-base position.

Actually, it is not necessary to decompose the cap into submerged and unsubmerged parts and consider the weight vectors E and H of the two parts separately. As Fig. 7(b) shows, the single weight vector J of the entire cap is equal in magnitude and opposite in direction to the buoyancy vector B. These two vectors have different lines of action, one through the cap’s center of mass and the other through its sphere-center. The two forces have a zero resultant, but
constitute a counterclockwise couple of magnitude \( Jd \), where \( J \) is the magnitude of the weight vector (i.e., the weight of the cap) and \( d \) is the horizontal distance between the two vectors.

In Naval Architecture, \( d \) is called the righting arm\(^7\)–\(^9\) of the tilted cap and its value depends on the amount of the tilt. The righting moment is the righting arm multiplied by the weight of the cap. For a ship, a graph of the righting arm versus its rolling tilt angle (called the ship’s heel angle) is known as the ship’s righting arm curve and is a basic consideration in ship design because it determines the restoring torque for various heel angles.

VI. CONCLUDING REMARKS

We again want to emphasize that the purpose of this paper is not to criticize Archimedes for not using or formulating results and concepts that arose centuries after his time. But just as it is important to point out consequences of the mechanics of Newton that are quite different under the mechanics of Einstein, it is also important to point out consequences of the mechanics of Archimedes that are quite different under the mechanics of Newton. For example, under the mechanics of Newton, it is possible for a system of five point particles to escape to infinity in a finite amount of time.\(^22\),\(^23\) Of course, this amazing result is not possible under the Theory of Relativity and it is important to point this out to a general audience. Similarly, it is important to point out that while under the mechanics of Archimedes a body, such as in Fig. 6, will always float stably in a horizontal-base position, this is not true under the mechanics of Newton.

Although Archimedes’ statement of Proposition 8 is correct for a cap of uniform density in a central field, this result should not be considered as intuitively obvious. It is not unreasonable to suspect (in the absence of a proof) that for some combination of relative density, relative radii of Earth and cap, and height of the base of the cap that the two horizontal-base positions of a cap are unstable, even in a uniform gravitational field. Indeed, in Book II of *On Floating Bodies* (see Ref. 5, pp. 263–300), Archimedes proved that a floating paraboloid can float stably in a nonvertical position with its base above the liquid on a flat Earth for certain combinations of relative density and ratios of height to diameter.\(^24\),\(^25\)

Another (contemporary) reason to be wary of Archimedes’ proposition is the fact that while a cap of uniform density in a circular orbit about Earth has two equilibrium configurations with its base horizontal, neither is a stable equilibrium. It can be shown that under the mechanics of Newton the only stable equilibrium configuration of an orbiting cap is one in which a line through its center of mass parallel to its base passes through the geocenter. Figure 8(a) shows the configuration for a hemispherical satellite.

The relevant theory from the classical mechanics is that the weight vector on a body in a central gravitational field exerts a torque about its center of mass that tends to orient the body so that the line through the center of mass about which the body’s moment of inertia is a minimum passes through center of the central field.\(^26\)–\(^33\) For a cap, this line of minimum moment of inertia is any line through the center of mass that is parallel to the base of the cap.\(^34\)

The consequence of this torque on the proof of Proposition 8 is that the line of action of the entire gravitational force on the cap shown in Fig. 3(a) will lie to the right of \( \Xi \), its center of mass. If it fell sufficiently to the right to coincide with the vertical line, then the resulting orientation will be an equilibrium one in which the base is tilted.

Even if the line of action of the weight vector falls between the center of mass and the sphere-center and the body is released with part of it submerged, it is possible for the cap to begin rotating clockwise about its center of mass rather than counterclockwise, as seen in Fig. 8(b). This is because Archimedes did not specify how deep the cap was to be submerged upon release in the liquid. Consequently, if it is released with only a tiny portion submerged, then the vertical buoyancy force will be very small and the counterclockwise torque that it produces around the cap’s center of mass could be less than the clockwise torque produced by the weight vector \( W \). But using Archimedes’ argument, \( W \) would pass through the center of mass and consequently produce no torque. The body would then only be subject to the counterclockwise torque of the buoyancy force.
In addition to being correct, the first part of our proof is simpler than Archimedes’ proof. This correct proof does not require any concepts that Archimedes was not familiar with, and in fact it doesn’t even require his Postulate 7 or Proposition 8 of *On the Equilibrium of planes I* (Secs. III and IV of the supplementary material).

The details of how a rigid floating body behaves in a liquid are still discussed at length today, even on a flat Earth, much less a spherical Earth.

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**APPENDIX A: A LEVER ON A SPHERICAL EARTH**

Although the centers of mass and gravity are trivially the same for a point particle in a central gravitational field, the differences between the two centers become apparent even for a rigid body consisting of just two point particles. For example, consider a rigid body in the form of a lever consisting of two particles connected by a straight massless rod. We take the mass of both particles to be the same, so that the center of mass of the lever is at the middle of the connecting rod.

Figure 9 shows a lever that is free to rotate about its center of mass (the fulcrum of the lever), which is held fixed at a distance of 2 units from the geocenter of a spherical Earth. We also take the length of the connecting rod to be 2 units. We take the central gravitational field to have constant magnitude, so that the gravitational force on each particle does not vary with its distance from the geocenter. The magnitude of the gravitational force (the scalar weight of each particle) is proportional to its mass, and we can take their weights to be 1 in suitable units. In Fig. 9, the weight vector of each particle is shown as a dashed vector of fixed magnitude with a line of action from each particle to the geocenter.

Next, we take the vertical direction to be along the line from the center of mass of the lever to the geocenter. Because the two weight vectors pass through the geocenter, so will their resultant. Using the parallelogram law of the addition of force vectors and basic trigonometry, we find that the line of action of the resultant vector will pass through the fulcrum as shown in Fig. 9(a). The magnitude of the resultant is 1.789 units, which is then the (scalar) weight of the lever in the particular orientation in Fig. 9(a).

Notice that there is no point along the line of action of the weight vector at which we can concentrate the two particles to yield the same resultant weight vector of magnitude 1.789. This is because the concentrated weight of the two particles that constitute the lever is 2 units regardless of where we place the one concentrated particle. This is generally true in a non-uniform gravitational field: there is (in general) no point on the line of action of the weight vector where we can concentrate the mass of the rigid body so that the weight vector of the concentrated mass is the same as the weight force vector of the rigid body. Only in a uniform gravitational field can we be assured that there is such a point, in which case it coincides with its center of mass.

Figure 9(b) shows the situation when the lever is rotated clockwise so that it is tilted at 45°. Again, using basic trigonometry and the parallelogram law, we find that the resultant gravitational force on it is a vector of magnitude 1.859 that passes through the geocenter at an angle of 7.02° from the vertical. The weight of this rotated lever is thus different from its weight in the position in Fig. 9(a). This is another property of non-uniform gravitational fields, that when a rigid body is rotated about its center of mass (or any other point), its weight (the magnitude of the weight vector) can vary.

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Fig. 9. Three orientations of a lever in a central gravitational field of constant magnitude.
Notice also, and most importantly for our discussion, that the line of action of the weight vector in Fig. 9(b) does not pass through the center of mass of the lever. We then know, from the classical mechanics, that there will be a torque exerted on the lever tending to rotate it about its center of mass. In Fig. 9(b), the torque is in the clockwise direction tending to rotate the lever arm to the vertical direction.

Figure 9(c) shows a configuration of the lever in which the weight vectors of the two masses pass through the fulcrum vertically, as does the resultant weight vector. The three vectors are shown slightly displaced in the figure for clarity. This is a stable equilibrium configuration and the weight of the lever is 2 units.

In summary, for a rigid body in a central gravitational field (or, more generally, in any nonuniform field):

(1) The line of action of its weight vector does not generally pass through its center of mass.
(2) There is generally no point on the line of action of its weight vector where its entire mass can be concentrated such that the weight vector of the concentrated mass is the same as the weight vector of the rigid body.
(3) There is no fixed point through which all lines of action pass as the body is translated or rotated.
(4) The weight of the body (the magnitude of its weight vector) depends on the position and orientation of the body.

What little remains of the concept of the “center of gravity” of a rigid body in Archimedes’ universe is that for each position and orientation of the rigid body there is a weight vector that generally does not pass through the body’s center of mass and whose magnitude and direction change as the body is moved or rotated. This weight vector is a sliding vector because it can be slid along its line of action; that is, it has no particular point of application along its line of action (see “Principle of Transmissibility,” Ref. 19, p. 38).

APPENDIX B: COUNTEREXAMPLE TO ARCHIMEDES’ PROOF (MATHEMATICAL DETAILS)

To simplify some later calculations, we take the density of the hemisphere outside of the ring in Fig. 6 to be zero and take the thickness of the ring to be zero. In addition, we take the density of the liquid to be infinite. It is as if the liquid has taken the thickness of the ring to be zero. In addition, we take the density of the liquid to be infinite. It is as if the liquid has no particular point of application along its line of action (see “Principle of Transmissibility,” Ref. 19, p. 38).

This is a stable equilibrium position of the hemisphere in which it is tilted at some angle \( \theta \) greater than zero and the base is above the liquid. To show this, it is more convenient to consider the potential energy of the cap rather than the forces acting on it. (This, of course, was not a tool in Archimedes’ tool chest.) If we consider the central force of Earth to be one in which the strength varies with distance as an inverse \( n \)th power with \( n \geq 0 \), then the potential energy of a point particle will be proportional to \( 1/d^{n-1} \) (for \( n \neq 1 \)) or \( \ln d \) (for \( n = 1 \)), where \( d \) is the distance of the particle from the geocenter. Since all of the mass of the hemisphere is concentrated in the ring, we can integrate around the ring to find the total potential energy \( PE(\theta) \) of the ring as a function of the tilt angle \( \theta \). The resulting integral is (up to a multiplicative constant that we set equal to one)

\[
PE(\theta) = \begin{cases} 
\int_{0}^{\pi} \left(A(\theta) - B(\theta)\cos \phi \right)^{1/n} d\phi, & n \neq 1 \\
\int_{0}^{\pi} \ln \left(A(\theta) - B(\theta)\cos \phi \right) d\phi, & n = 1, 
\end{cases} \tag{A1}
\]

where

\[
A(\theta) = R_{0}^{2} + R_{1}^{2} - 2R_{0}R_{1}\cos \theta \cos \beta \tag{A2}
\]

and

\[
B(\theta) = 2R_{0}R_{1}\sin \theta \sin \beta. \tag{A3}
\]

The horizontal-base position \( \theta = 0 \) is an equilibrium position, since the first derivative of the potential energy is zero there \( (PE'(0) = 0) \). The second derivative test states that it is an unstable equilibrium position (a local maximum of
It can be shown that for every value of $R_0/R_1 > 1$ and every value of $n \geq 0$, the values of $\beta$ for which this inequality is satisfied lie in an interval $(\beta_{\min}, 90^\circ)$, where $\beta_{\min}$ is the root of the left-hand expression in the interval $(0, 90^\circ)$. It can further be shown that for any specified tilt angle $\theta$ strictly between $0^\circ$ and $90^\circ$, some value of $\beta$ in that range will cause the hemisphere to be in stable equilibrium at that tilt angle.

Notice that for any fixed $R_1$ and $n$, the value of $\beta_{\min}$ approaches $90^\circ$ as $R_0$ approaches infinity (that is, as Earth becomes flatter). Consequently, for a flat Earth ($R_0 = \infty$), the horizontal-base position is stable for all $R_1$, $\beta$, and $n$. This example thus emphasizes the difference between the behavior of such a cap in a uniform and a central gravitational field.

As an explicit example, this inequality is satisfied for a hemisphere with $R_0 = 8$, $R_1 = 2$, and $\beta = 85^\circ$ in Archimedes’ universe ($n = 0$), causing the horizontal-base position to be unstable. To find the corresponding stable tilt position, we can examine the potential energy diagram of the hemisphere for all tilt angles between $-90^\circ$ and $90^\circ$ [see Fig. 11(a)]. It can be seen that there are two local minima at $\theta = \pm 45.6^\circ$, corresponding to two symmetrically situated stable tilt angles of the hemisphere. Figure 11(b) shows the configuration of the hemisphere for $R_0 = 3$ and $\theta = 45.6^\circ$. The same hemisphere in an inverse-square central field (Newton’s universe of $n = 2$) will float stably at a tilt angle of $\theta = 76.2^\circ$.

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$PE(\theta)$ if $PE''(\theta) < 0$ and an asymptotically stable position (a local minimum of $PE(\theta)$) if $PE''(\theta) > 0$. By evaluating this second derivative, we find that the horizontal-base position is unstable if

$$
(3 - n) \cos^2 \beta - 2 \left( \frac{R_0}{R_1} + \frac{R_1}{R_0} \right) \cos \beta + (1 + n) > 0.
$$

(A4)

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Fig. 11. (a) Potential energy of a hemisphere as a function of its tilt angle $\theta$. (b) Diagram of the hemisphere in its stable tilted position.

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1A Greek manuscript dating from about the ninth century and containing both books of On Floating Bodies was translated into Latin by the Flemish Dominican William of Moerbeke in 1269. Traces of the Greek manuscript were lost in the 14th century, but Moerbeke’s holograph remains intact in the Vatican library (Codex Ottobonianus Latinus 1850) (Ref. 2). Moerbeke’s Latin translation was the source of all versions of On Floating Bodies from his time until the 20th century. Moerbeke’s translation of Book I of On Floating Bodies was first printed in 1543 by Niccolò Fontana Tartaglia in Venice. Books I and II were first printed in 1565, independently by Curtius Trojanus in Venice and by Federico Commandino in Bologna (Ref. 3). A palimpsest (Refs. 14–16) from the tenth century, discovered and edited by Heiberg (Ref. 13) in 1906, contains the only extant Greek text. The texts by Dijksterhuis (Ref. 4) and Heath (Ref. 5) contain the only translations/paraphrases of On Floating Bodies presently available in English, although both have very incomplete treatments of Propositions 8 and 9 of Book I.


12See supplementary material at http://dx.doi.org/10.1119/1.4943660 for a PDF containing translations and comments for three of Archimedes’ propositions and postulates, and a figure that supplements Fig. 4.

13J. L. Heiberg, Archimedis Opera Omnia, with corrections by E. S. Stamatis (B. G. Teubner, Stuttgart, 1972), Vol. II.


17Federico Commandino, Archimedes: De Iis Quae Vehuntur in Aquis, Libri Duo (Benacius, Bologna, 1565).

This torque on a body is an example of a gravity-gradient torque, a phenomenon that has been known since Newton’s time. For a rigid body with rotational symmetry in Archimedes’ universe, an approximate formula for the gravity-gradient torque $T$ is $T \approx \left( W/2R_0 \right) \left( r_{\text{max}}^2 - r_{\text{min}}^2 \right) \sin 2\theta$, where $r_{\text{max}}$ is the body’s minimum radius of gyration, assumed to be about its axis of rotational symmetry, and $r_{\text{max}}$ is the body’s maximum radius of gyration, which is about any axis perpendicular to the axis of rotational symmetry through the body’s center of mass. Additionally, $W$ is the weight of the body if its entire mass were concentrated at its center of mass, $R_0$ is the distance from the geocenter to the body’s center of mass, and $\theta$ is the clockwise angle from the vertical to the body’s axis of rotational symmetry. This formula is an excellent approximation if $R_0$ is much larger than the dimensions of the solid. In his Principia Mathematica (Book III, Proposition 39, Ref. 18), Newton showed that the gravity-gradient torque exerted by the sun on the moon causes the precession of the equinoxes (Refs. 27–32). Later, the gravity-gradient torque exerted by Earth on the moon was shown to explain why the moon always points the same face toward Earth, a phenomenon called tidal locking (Refs. 28–32).

Gravity-gradient torques have been exploited since the late 1960s to keep the axes of minimum moment of inertia of artificial near-Earth satellites always pointing towards Earth (Ref. 33).

Because Earth’s equatorial bulge, its rotational axis is its axis of maximum moment of inertia and any line through the geocenter perpendicular to its rotational axis is an axis of minimum moment of inertia. The central gravitational fields of the sun and moon exert separate gravity-gradient torques tending to orient Earth so that its equatorial plane passes through the sun or moon. Then, because the moon and sun revolve about Earth (relative to an observer on Earth), the effect of the two torques is to cause Earth’s rotational axis to slowly precess about a line perpendicular to the ecliptic plane with a period of about 26,000 years.


For a hemisphere of uniform density and radius $r$, the center of mass is along its axis of symmetry a distance $3r/8$ from its base, a fact first demonstrated by Archimedes (The Method, Ref. 5, p. 27). The maximum moment of inertia through the hemisphere’s center of mass is about its axis of symmetry and its minimum moment of inertia through its center of mass is about any line parallel to its base. The corresponding moments are $I_{\text{max}} = \frac{2mr^2}{5}$ and $I_{\text{min}} = \frac{83mr^2}{120}$, where $m$ is the mass of the hemisphere. For the lever in Appendix A, its moment of inertia about the line through its two point masses is zero, and so this line is its axis of minimum moment of inertia. Thus, as Fig. 9(e) illustrates, the lever will be in stable equilibrium when this line passes through the geocenter.


This instrument is listed at $16.65 in the 1909 Central Scientific catalogue. The taps in front allow the experimenter to use 20, 40, 80 or 160 turns in the coil, thus giving four current ranges. And, for heavy currents the brass ring on which the coils are wound can be used by itself, although the sensitivity is greatly decreased. This is a brass and mahogany instrument, more typical of nineteenth century practice. The device is in the Greenslade Collection. (Notes and picture by Thomas B. Greenslade, Jr., Kenyon College)