CS 430/536
Computer Graphics I

NURBS Drawing

Week 5, Lecture 10

David Breen, William Regli and Maxim Peysakhov
Geometric and Intelligent Computing Laboratory
Department of Computer Science
Drexel University
`http://gicl.cs.drexel.edu`

Drexel UNIVERSITY

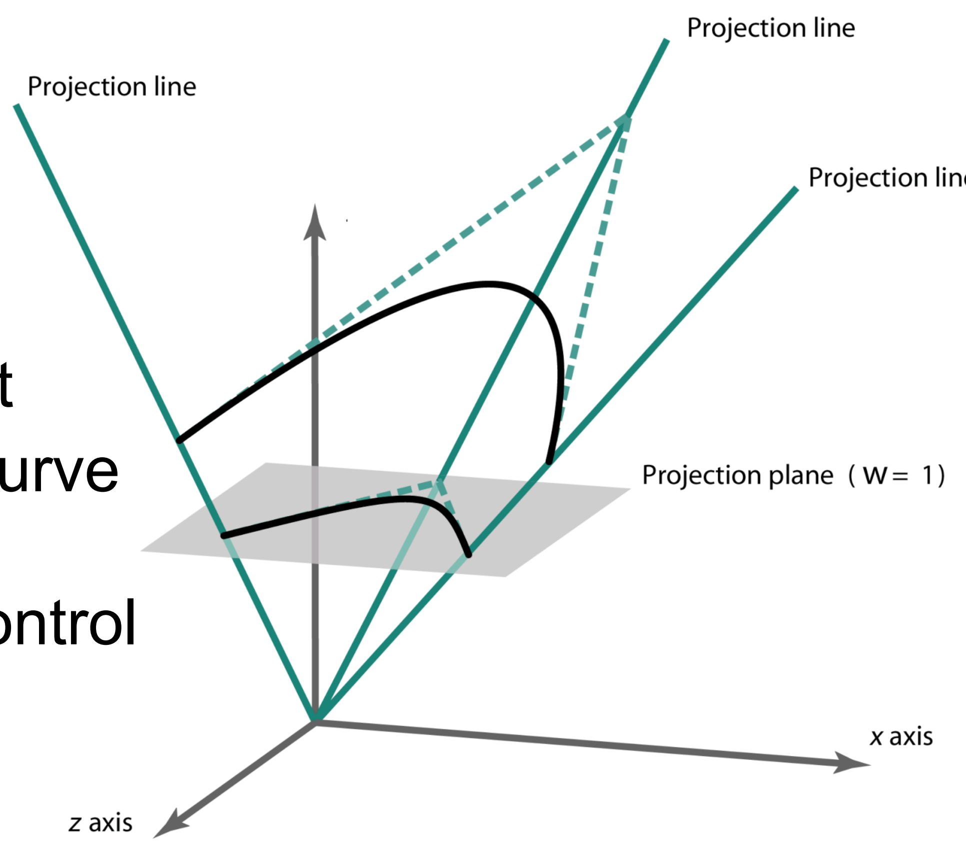
GICL

Outline

- Conic Sections via NURBS
- Knot insertion algorithm
- The de Boor's algorithm
  - for B-Splines
  - for NURBS
- Oslo Algorithm
- Barycentric Coordinates
- Discussion of homework #3

Conic Sections via NURBS

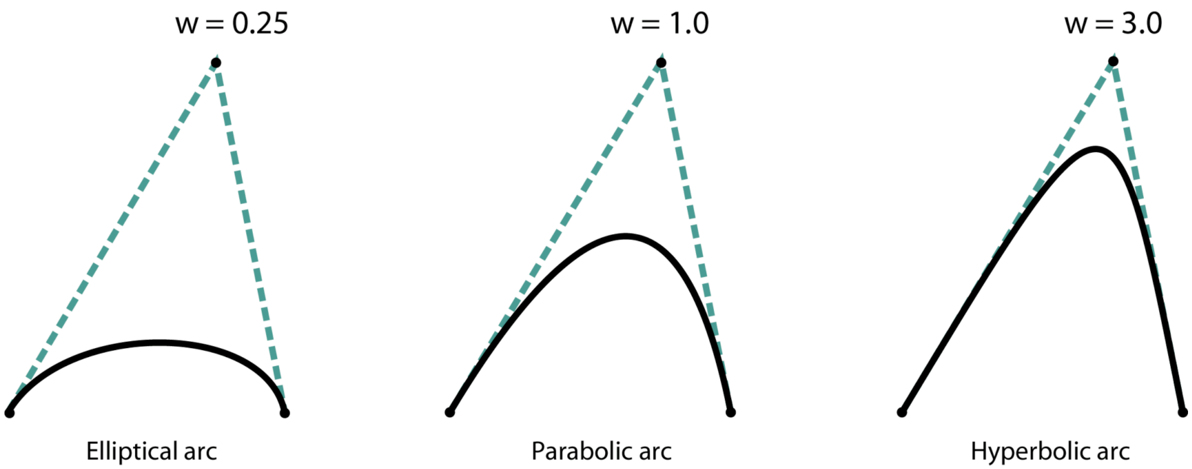
- Obtained via projection of the 3D parabola onto a plane
- Note:
  - 3D Case: rational curve is a 4D object
  - 2D Case: rational curve is a 3D object
  - assign *w* to each control point



From http://devworld.apple.com/dev/techsupport/develop/issue25/schneider.html

Conic Sections via NURBS

- Define the curve with three control points
- Weights of first/last control points are 1
- For center control point
  - w<1 gives an ellipse
  - w>1 gives a hyperbola
  - w=1 gives a parabola

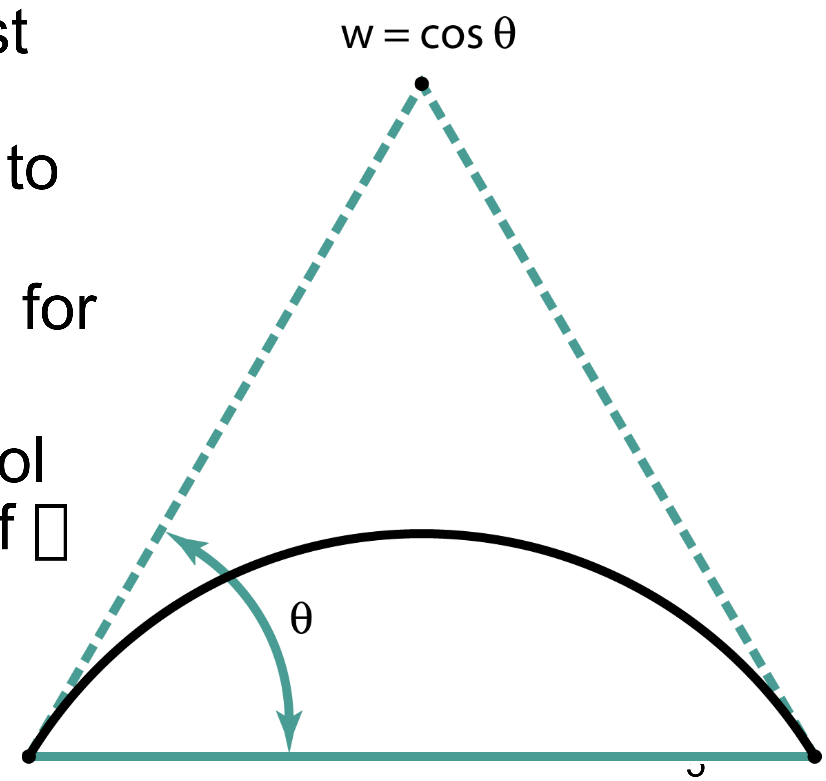


  - Knot vector is {0.0, 0.0,0.0, 1.0, 1.0, 1.0}

From http://devworld.apple.com/dev/techsupport/develop/issue25/schneider.html

Conic Sections via NURBS: A Circular Arc

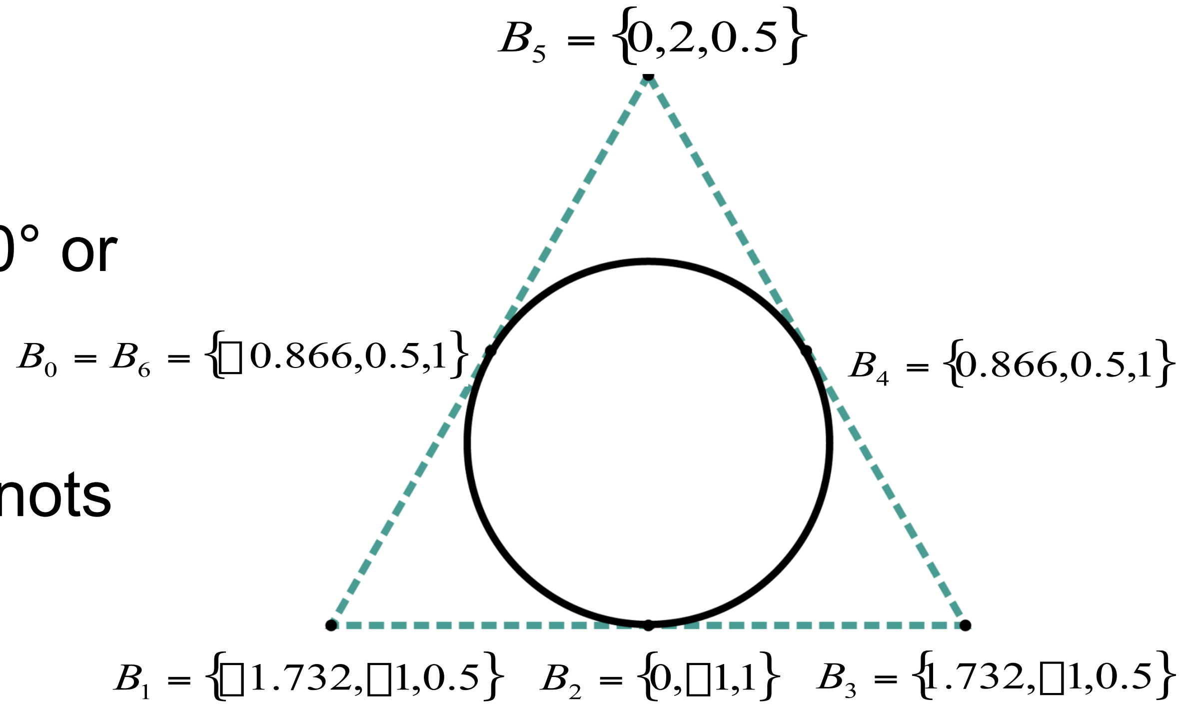
- The two sides of the control polygon are of equal length
- The chord connecting the first and last control points meets each leg at an angle θ equal to half the angular extent of the desired arc (for instance, 30° for a 60° arc)
- The weight of the inner control point is equal to the cosine of θ
- Knot vector is {0.0, 0.0, 0.0, 1.0, 1.0, 1.0}



From http://devworld.apple.com/dev/techsupport/develop/issue25/schneider.html

Conic Sections via NURBS: A Circle

- What if we need an arc of >180°?
- Idea:
  - Use multiple 90° or 120° arcs
  - stitch them together with knots



Example:
   3 arcs of 120°

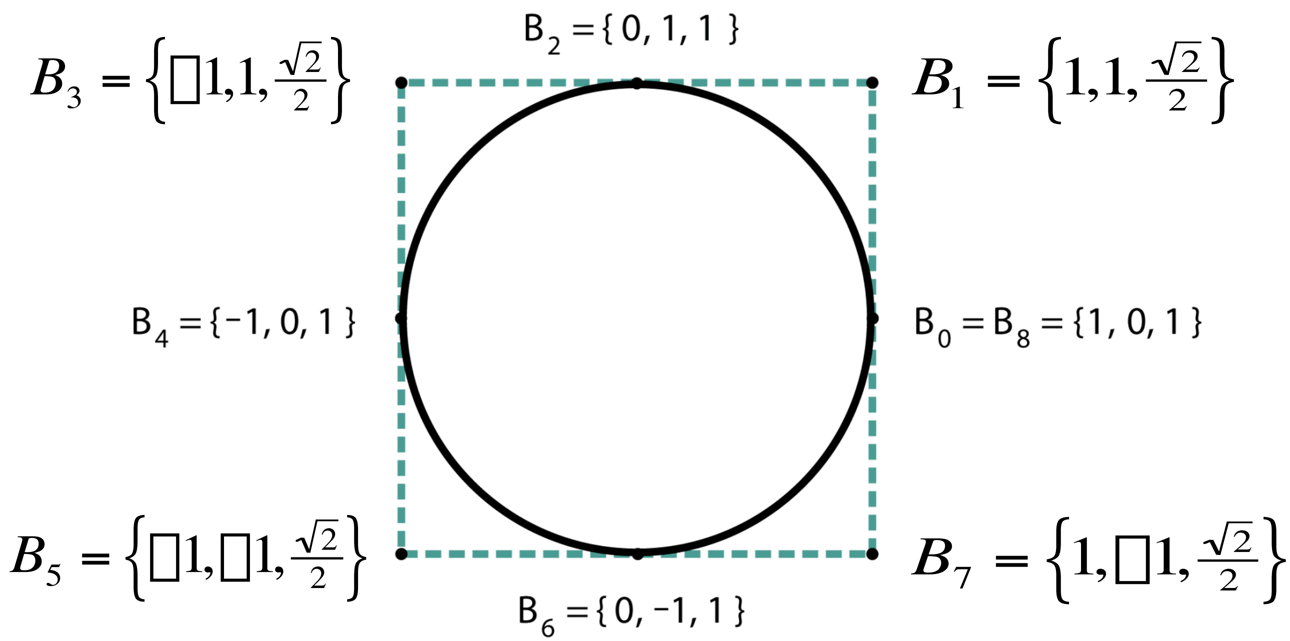
$$\text{knots} = \left\{ 0,0,0,1,1,2,2,3,3,3 \right\}$$

From http://devworld.apple.com/dev/techsupport/develop/issue25/schneider.html

Conic Sections via NURBS

Example:

4 arcs of 90°

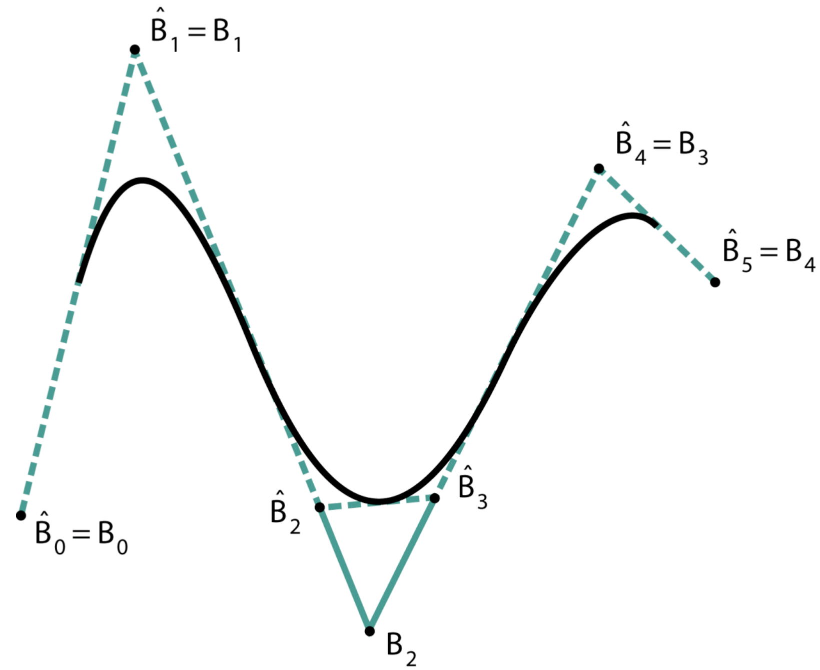


$$\text{knots} = \left\{0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, 1, 1, 1\right\}$$

From http://devworld.apple.com/dev/techsupport/develop/issue25/schneider.html

Knot Insertion

- Issue: More control points mean more control
- How do we add more points and keep same curve?

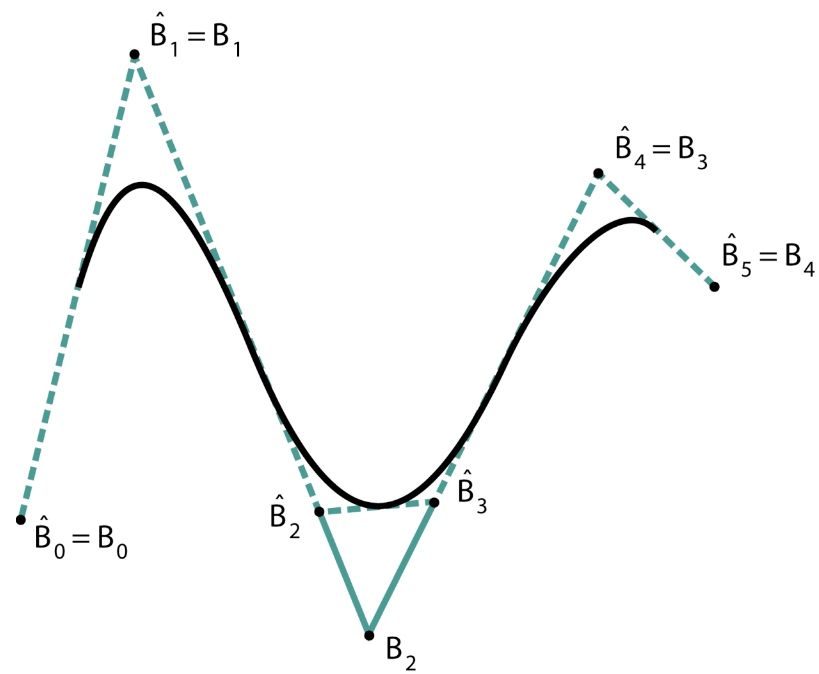


From http://devworld.apple.com/dev/techsupport/develop/issue25/schneider.html

Knot Insertion

- Basic Approach
  - Decide where we'd like to tweak the curve
  - Add a new knot
  - Find affected *d-1* control points
  - Replace it with *d* new control points

Example:
New knot at u=2.6



From http://devworld.apple.com/dev/techsupport/develop/issue25/schneider.html

Knot Insertion

- Given: $n+1$ control points $(P_0, P_1, ..., P_n)$, a knot vector of $m+1$ knots $U = \{ u_0, u1_, ..., u_m \}$ and a degree d B-spline curve $C(u)$.
- Insert a new knot t into the knot vector without changing the shape of the curve.
- If t lies in knot span $[u_k, u_{k+1})$, only the basis functions for $(P_k, \ldots P_{k-d})$ are non-zero.
- Find d new control points Q_k on edge $P_{k-1}P_k$, Q_{k-1} on edge $P_{k-2}P_{k-1}$, ..., and Q_{k-d+1} on edge $P_{k-d}P_{k-d+1}$
- All other control points are unchanged.
- Note that $d-1$ control points of the original control polyline are removed and replaced with d new control points.

See http://www.cs.mtu.edu/~shene/COURSES/cs3621/NOTES/spline/B-spline/single-insertion.html

Knot Insertion Algorithm

- Create new control point

$$Q_j = (1 - \alpha_j)P_{j-1} + \alpha_j P_j$$

- Where α is defined as

$$\alpha_j = \frac{t - u_j}{u_{j+d} - u_j}$$

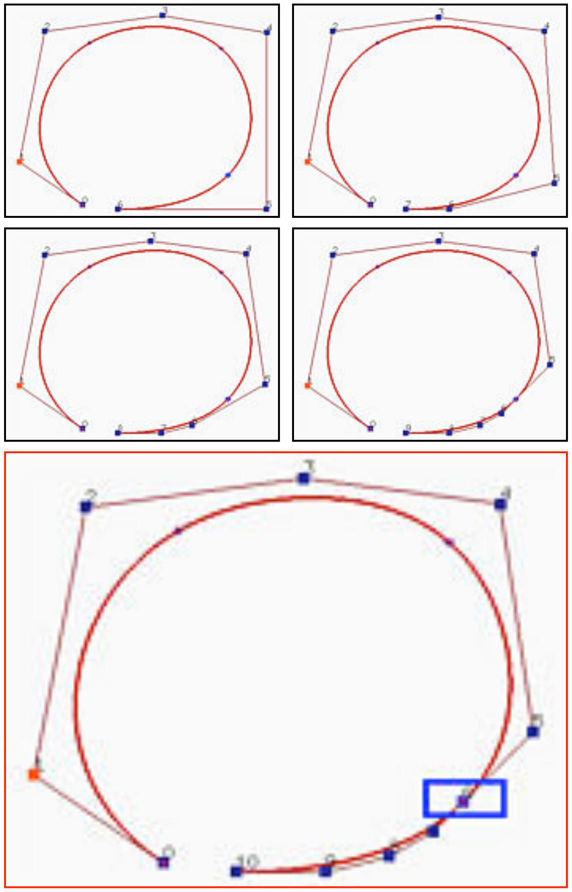
See http://www.cs.mtu.edu/~shene/COURSES/cs3621/NOTES/spline/B-spline/single-insertion.html

Properties of Knot Insertion

- Increasing the multiplicity of a knot decreases the number of non-zero basis functions at this knot
- At a knot of multiplicity d, there will be only one non-zero basis function
- Corresponding point on the curve $\mathbf{p}(u)$ is affected by *exactly one* control point $\mathbf{p}_i$
- In fact $\mathbf{p}(u)$ is $\mathbf{p}_i$!

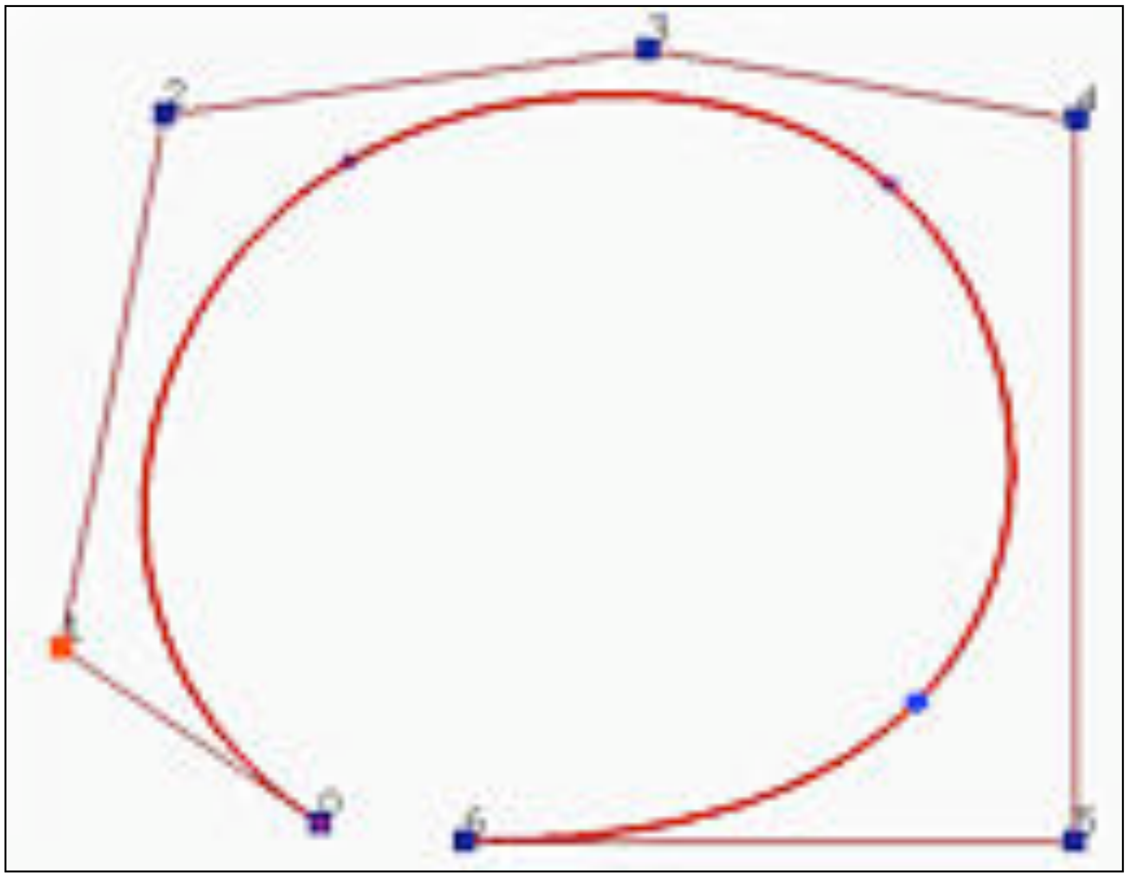
The de Boor Algorithm

- Generalization of de Casteljau's algorithm
- It provides a fast and numerically stable way for finding a point on a B-spline curve
- Observation: if a knot *u* is inserted *d* times to a B-spline, then p(u) is the point on the curve.
- Idea: We just simply insert *u* d times and the *last* point is **p**(*u*)!

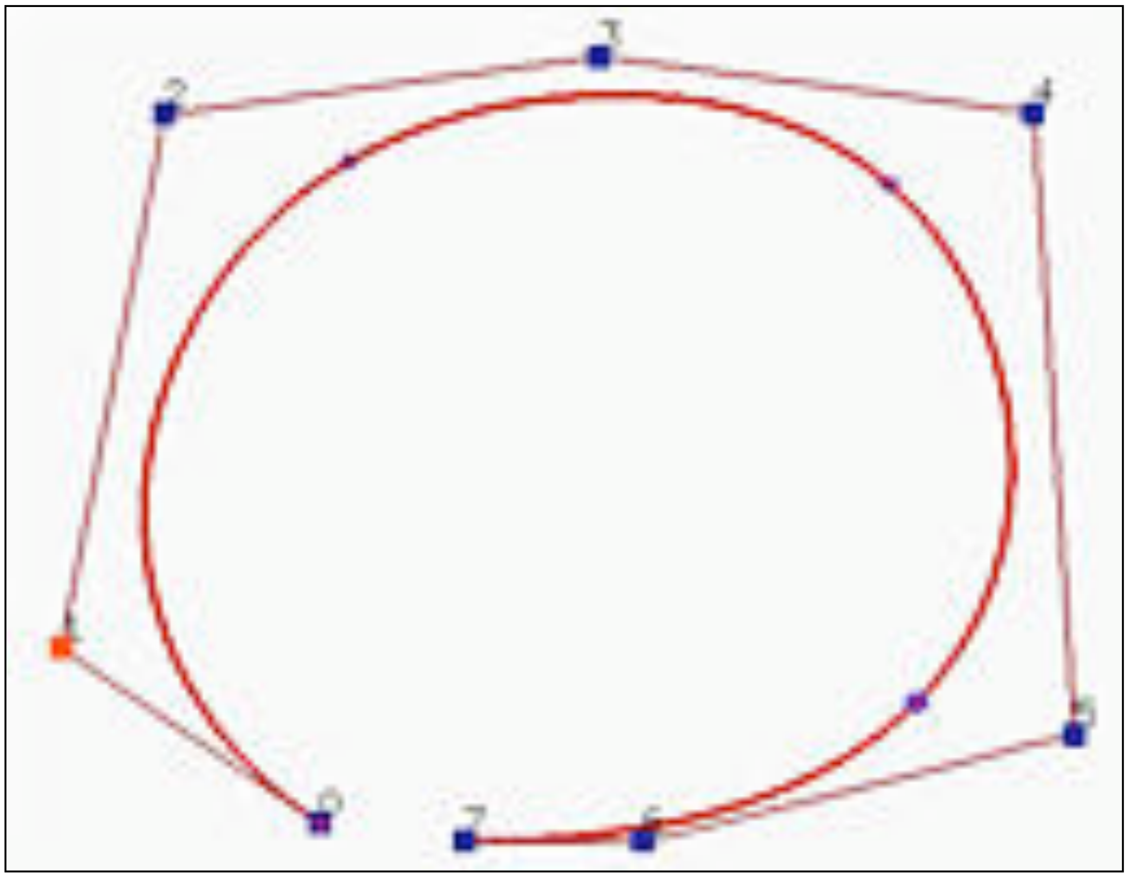


See http://www.cs.mtu.edu/~shene/COURSES/cs3621/NOTES/spline/B-spline/de-Boor.html

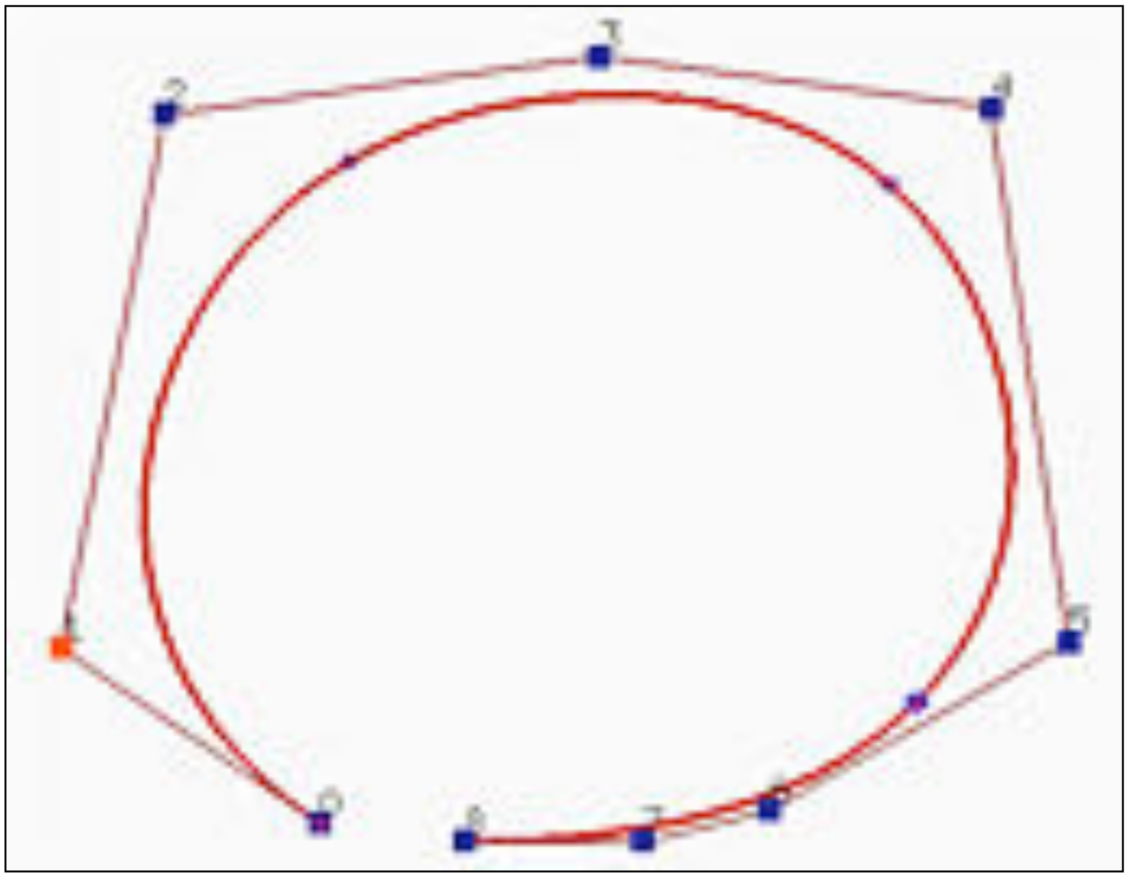
The de Boor Algorithm



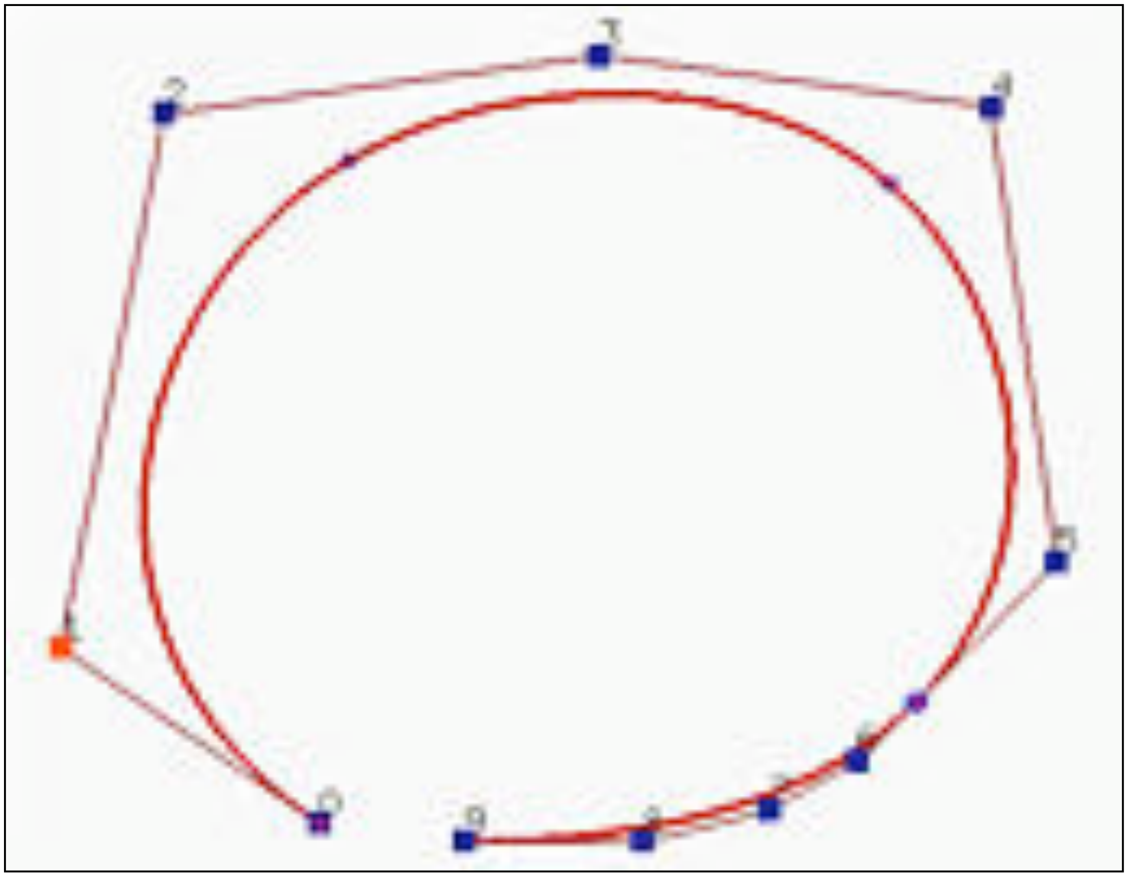
The de Boor Algorithm



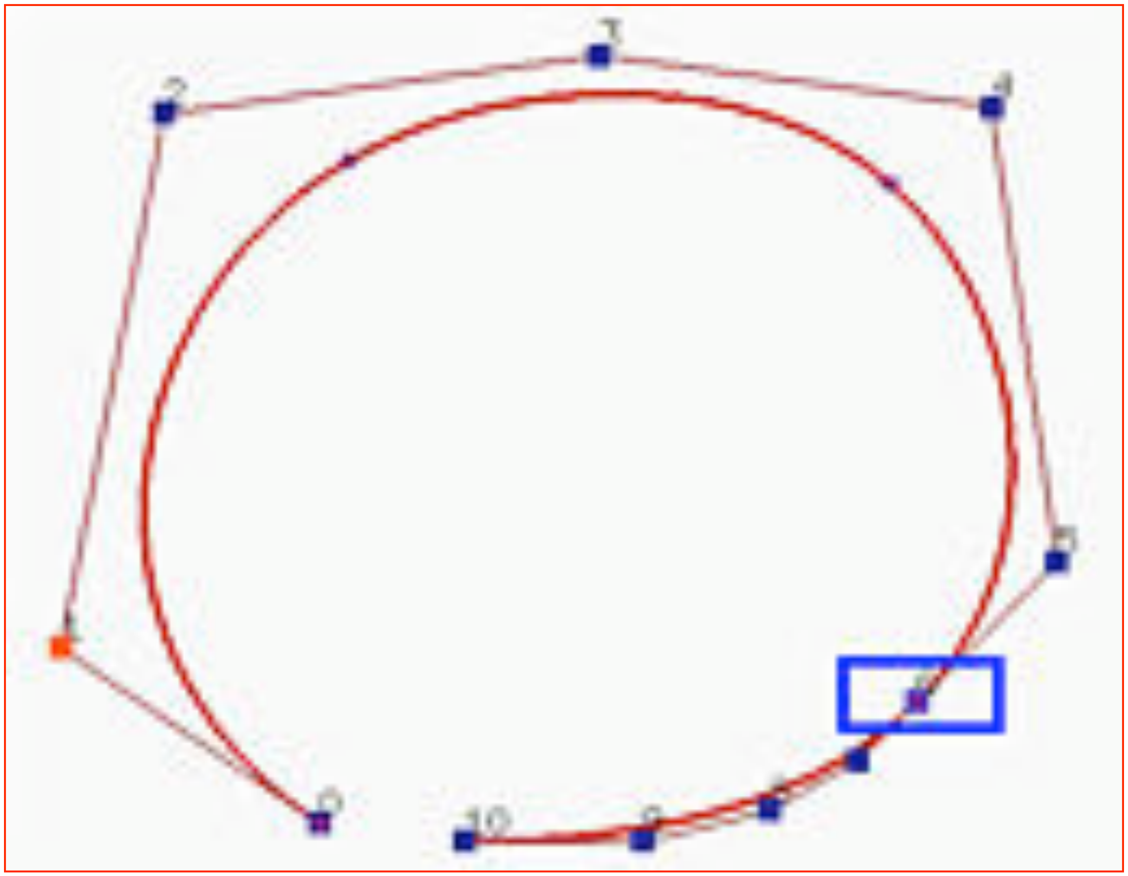
The de Boor Algorithm



The de Boor Algorithm



The de Boor Algorithm



De Boor's Algorithm

If u lies in $[u_k, u_{k+1})$ and $u \neq u_k$, let $h = d$

If $u = u_k$ and u_k is a knot of multiplicity s, let $h = d - s$

Copy the affected control points $\mathbf{p}_{k-s}$, $\mathbf{p}_{k-s-1}$,…, $\mathbf{p}_{k-d+1}$, $\mathbf{p}_{k-d}$
to a new array and rename them as $\mathbf{p}_{k-s,0}$, $\mathbf{p}_{k-s-1,0}$, …, $\mathbf{p}_{k-d+1,0}$

for r := 1 **to** h **do**

    for i := $k-d+r$ **to** $k-s$ **do**

    {

        Let $a_{i,r} = (\underline{u} - u_i) / (u_{i+d-r+1} - u_i)$

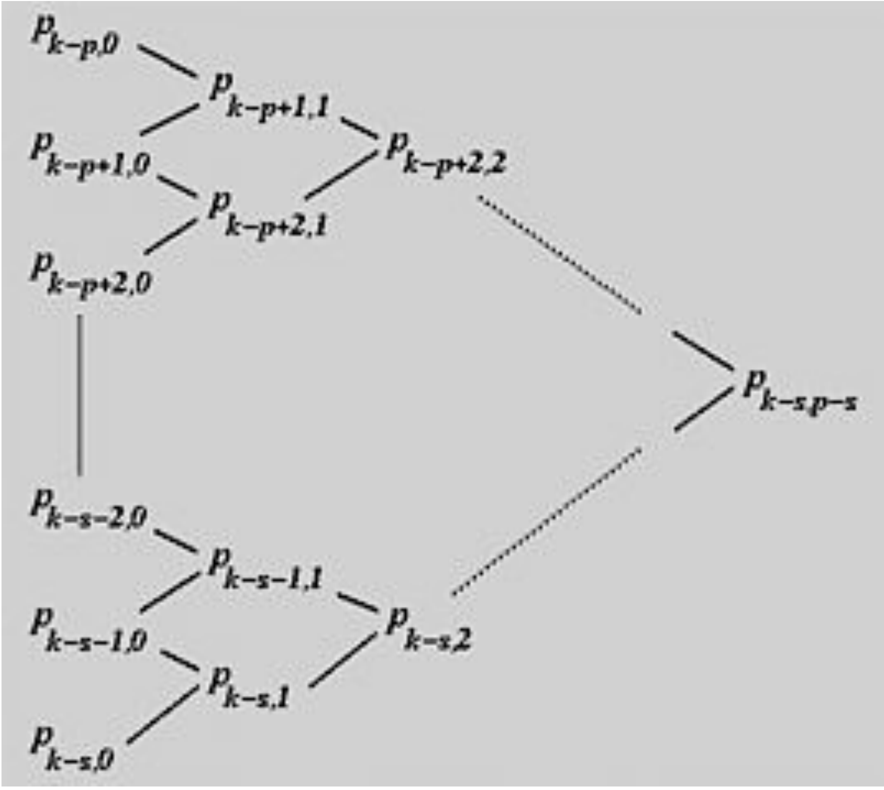
        Let $\mathbf{p}_{i,r} = (1 - a_{i,r})\, \mathbf{p}_{i-1,r-1} + a_{i,r}\, \mathbf{p}_{i,r-1}$

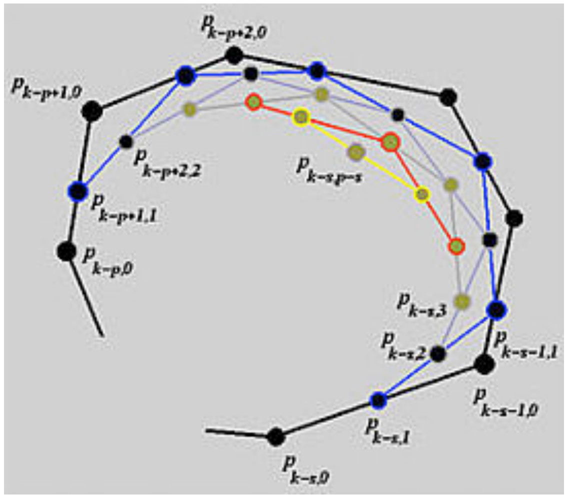
    }

$\mathbf{p}_{k-s,d-s}$ is the point $\mathbf{p}(u)$.

De Boor's Algorithm (*cont*)

for u := 0 **to** u_{max} **do**
{
  …
  for *r* := 1 **to** *h* **do**
    for *i* := *k-p+r* **to** *k-s* **do**
    {
      Let $a_{i,r} = (\underline{u} - u_i) / (u_{i+p-r+1} - u_i)$
      Let $\mathbf{p}_{i,r} = (1 - a_{i,r})\, \mathbf{p}_{i-1,r-1} + a_{i,r}\, \mathbf{p}_{i,r-1}$
    }
  $\mathbf{p}_{k-s,p-s}$ is the point $\mathbf{p}(u)$.
}





Compiled from Lecture notes of **Dr. Ching-Kuang Shene** @ Michigan Technological University

Example of de Boor's Algorithm

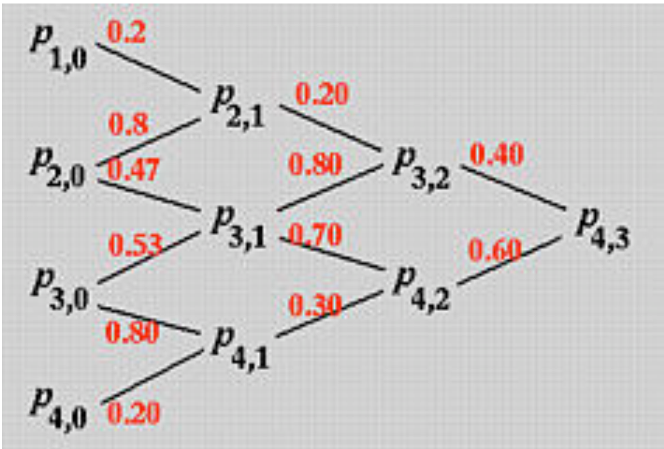
Degree 3 B-spline curve (*i.e.*, d = 3)

Defined by seven control points $\mathbf{p}_0, \ldots, \mathbf{p}_6$

And knot vector:

$u_0 = u_1 = u_2 = u_3$	u_4	u_5	u_6	$u_7 = u_8 = u_9 = u_{10}$
0	0.25	0.5	0.75	1

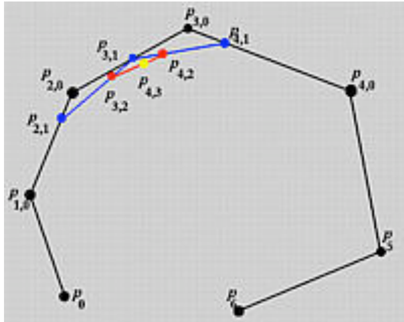
u = 0.4



$a_{4,1} = (u - u_4) / (u_{4+3} - u_4) = 0.2$
$a_{3,1} = (u - u_3) / (u_{3+3} - u_3) = 8/15 = 0.53$
$a_{2,1} = (u - u_2) / (u_{2+3} - u_2) = 0.8$
$\mathbf{p}_{4,1} = (1 - a_{4,1})\mathbf{p}_{3,0} + a_{4,1}\mathbf{p}_{4,0} = 0.8\mathbf{p}_{3,0} + 0.2\mathbf{p}_{4,0}$
$\mathbf{p}_{3,1} = (1 - a_{3,1})\mathbf{p}_{2,0} + a_{3,1}\mathbf{p}_{3,0} = 0.47\mathbf{p}_{2,0} + 0.53\mathbf{p}_{3,0}$
$\mathbf{p}_{2,1} = (1 - a_{2,1})\mathbf{p}_{1,0} + a_{2,1}\mathbf{p}_{2,0} = 0.2\mathbf{p}_{1,0} + 0.8\mathbf{p}_{2,0}$

$a_{4,2} = (u - u_4) / (u_{4+3-1} - u_4) = 0.3$
$a_{3,2} = (u - u_3) / (u_{3+3-1} - u_3) = 0.8$
$\mathbf{p}_{4,2} = (1 - a_{4,2})\,\mathbf{p}_{3,1} + a_{4,2}\mathbf{p}_{4,1} = 0.7\mathbf{p}_{3,1} + 0.3\mathbf{p}_{4,1}$
$\mathbf{p}_{3,2} = (1 - a_{3,2})\,\mathbf{p}_{2,1} + a_{3,2}\mathbf{p}_{3,1} = 0.2\mathbf{p}_{2,1} + 0.8\mathbf{p}_{3,1}$

$a_{4,3} = (u - u_4)/ (u_{4+3-2} - u_4) = 0.6$
$\mathbf{p}_{4,3} = (1 - a_{4,3})\mathbf{p}_{3,2} + a_{4,3}\mathbf{p}_{4,2} = 0.4\mathbf{p}_{3,2} + 0.6\mathbf{p}_{4,2}$



Compiled from Lecture notes of **Dr. Ching-Kuang Shene** @ Michigan Technological University

Similar but Different

De Casteljau's:

- Dividing points are computed with a pair of numbers *(1 - u)* and *u* that never change
- Can be used for curve subdivision
- Uses *all* control points

De Boor's

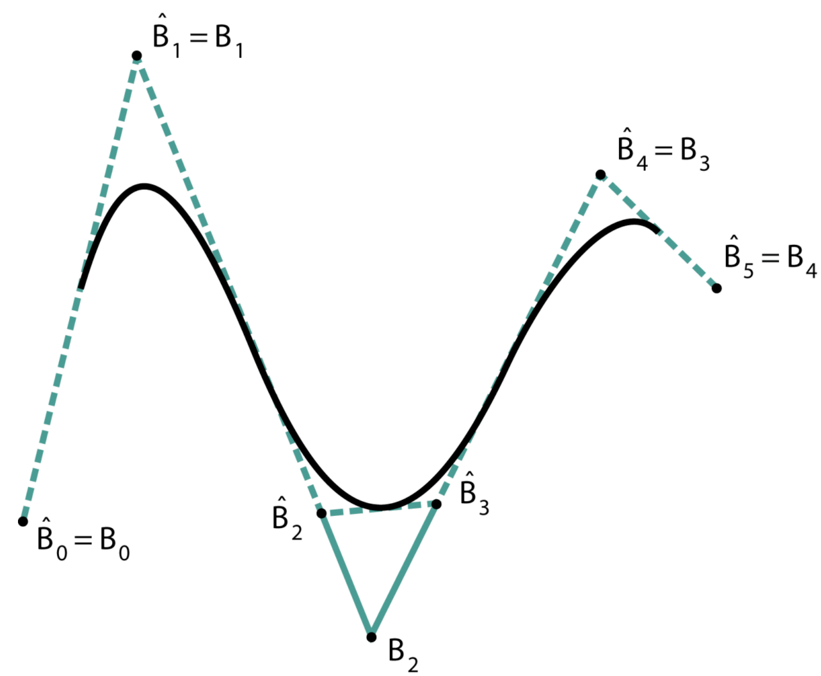
- These pairs of numbers are different and depend on the column number and control point number
- Intermediate control points are not sufficient
- *d+1* affected control points are involved in the computation

De Boor's: Curves

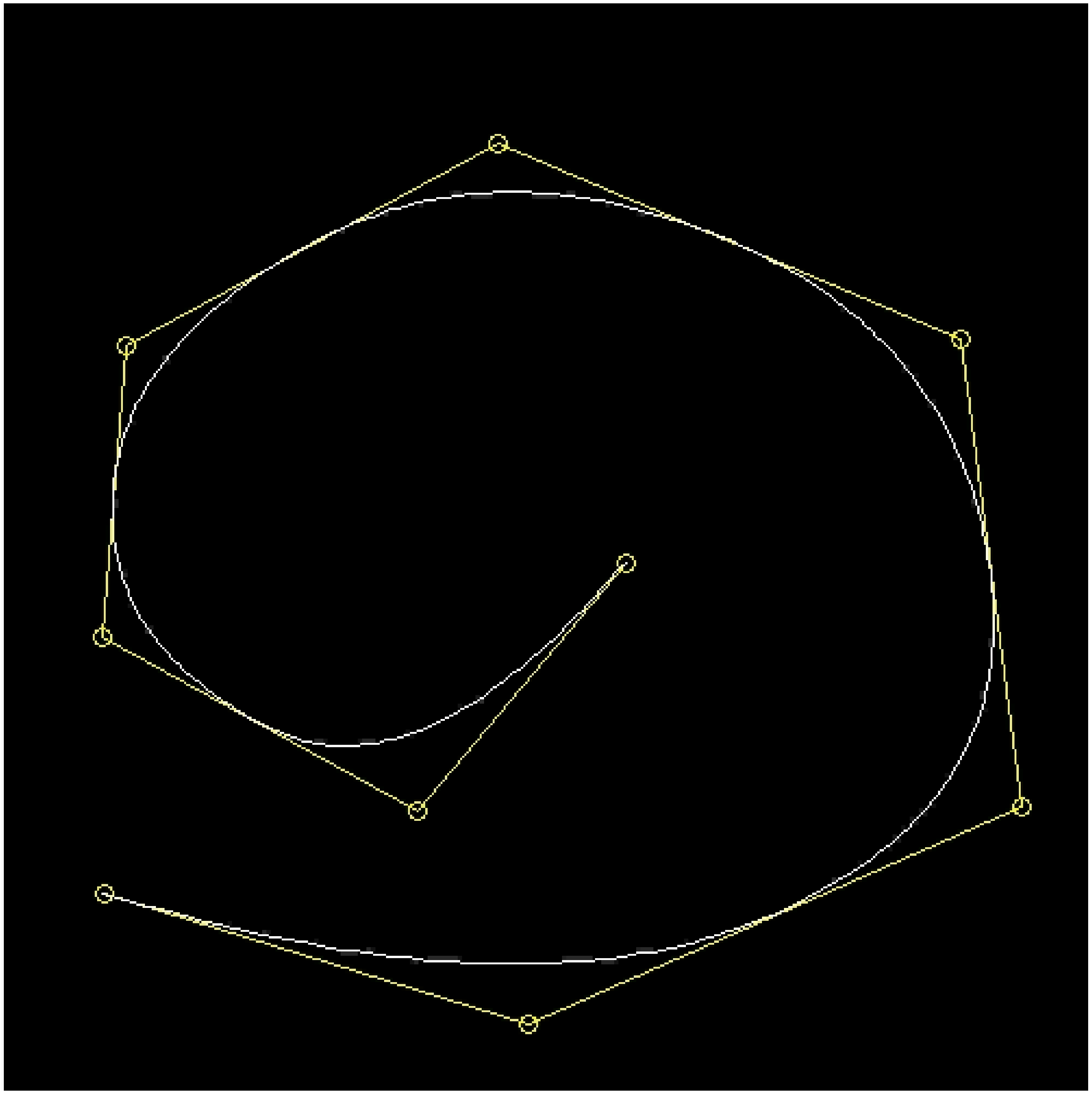
Animated by Max Peysakhov @ Drexel University

Oslo Algorithm

- A subdivision algorithm for B-splines, the basic idea:
- Take the curve with $m+1$ control points P_0 to P_m
- Insert a knot in any point (0.5 maybe?)
- As a result you will have 2 new points P_k' and P_k''
- Take curves with $m+1$ control points $P_0 \ldots P_k', P_k'' \ldots P_{m-1}$ and $P_1 \ldots P_k', P_k'' \ldots P_m$
- Apply procedure recursively on each part



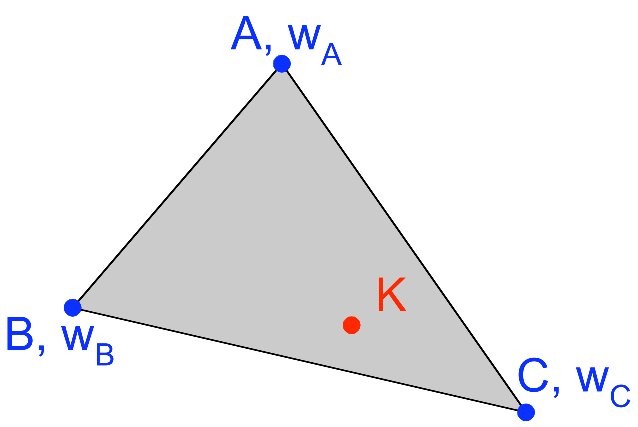
Oslo Algorithm



Recorded from: http://heim.ifi.uio.no/~trondbre/OsloAlgApp.html

Barycentric Coordinates

- By Ceva's Theorem:
  - For any point K inside the triangle ABC
  - Consider the existence of *masses* w_A, w_B, and w_C, placed at the vertices of the triangle
  - Their center of gravity (***barycenter***) will coincide with the point K.



- August Ferdinand Moebius (1790-1868) defined (1827) w_A, w_B, and w_C as the ***barycentric*** coordinates of K
- $K = w_A A + w_B B + w_C C$

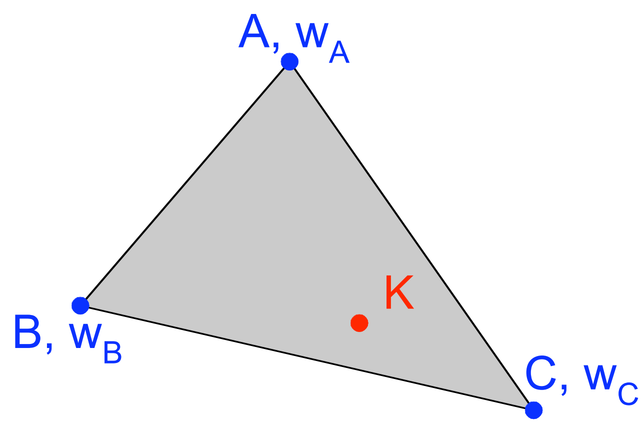
http://www.cut-the-knot.org/triangle/barycenter.shtml

Properties of Barycentric Coordinates

- Not unique
- Can be generalized to negative masses
- Can be made unique by setting

$$w_A + w_B + w_C = 1$$

- $w_A = 0$ for points on BC
  $w_B = 0$ for points on AC
  $w_C = 0$ on AB



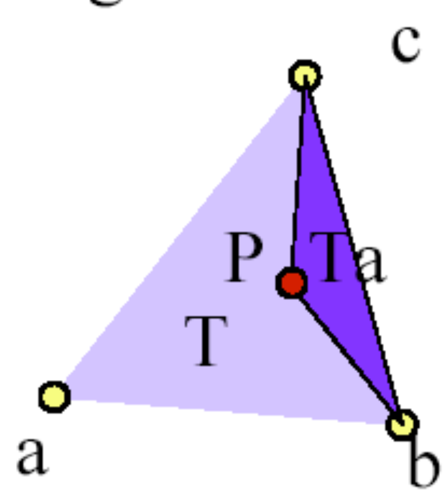
http://www.cut-the-knot.org/triangle/barycenter.shtml

Given P, how can we compute α, β, γ ?

- Compute the areas of the opposite subtriangle
  - Ratio with complete area

$$\alpha = A_a/A, \quad \beta = A_b/A \qquad \gamma = A_c/A$$

Use signed areas for points outside the triangle



Area Ta:
$|(b-P) \times (c-P)|/2$

Calculating the Weights

- Given vertices A, B, C and Centroid K
- What are the weights, w_A, w_B, w_C?

$$x_K = w_A x_A + w_B x_B + w_C x_C$$

$$y_K = w_A y_A + w_B y_B + w_C y_C$$

- Substitute $w_C = 1 - w_A - w_B$

$$x_K = w_A x_A + w_B x_B + (1 - w_A - w_B) x_C$$

$$y_K = w_A y_A + w_B y_B + (1 - w_A - w_B) y_C$$

Calculating Weights (cont.)

- Solve for w_A and w_B

$$w_A = \frac{(x_B - x_C)(y_C - y_K) - (x_C - x_K)(y_B - y_C)}{(x_A - x_C)(y_B - y_C) - (x_B - x_C)(y_A - y_C)}$$

$$w_B = \frac{(x_A - x_C)(y_C - y_K) - (x_C - x_K)(y_A - y_C)}{(x_B - x_C)(y_A - y_C) - (x_A - x_C)(y_B - y_C)}$$

- $w_C = 1 - w_A - w_B$

Onto…

- Bézier Surfaces
- B-spline Surfaces
- NURBS Surfaces
- Faceting, Subdivision, Tessellation
- 3D Objects

Programming assignment 3

- Input PostScript-like file containing polygons
- Output B/W XPM
- Implement viewports
- Use Sutherland-Hodgman intersection for polygon clipping
- Implement scanline polygon filling. (*You can not use flood filling algorithms*)