Transformations

CS 432 Interactive Computer Graphics
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Objectives

• Introduce standard transformations
  - Rotation
  - Translation
  - Scaling
  - Shear

• Derive homogeneous coordinate transformation matrices

• Learn to build arbitrary transformation matrices from simple transformations
General Transformations

A transformation maps points to other points and/or vectors to other vectors.

\[ Q = T(P) \]
\[ v = T(u) \]

\[ P \rightarrow Q \]

\[ u \rightarrow v \]
Affine Transformations

• Line preserving
• Characteristic of many physically important transformations
  - Rigid body transformations: rotation, translation
  - Scaling, shear
• Importance in graphics is that we need only transform endpoints of line segments and let implementation draw line segment between the transformed endpoints
Pipeline Implementation

T (from application program)

T(u)

T(v)

vertices → transformation → T(u) → rasterizer → frame buffer

T(v) → pixels

vertices
Notation

We will be working with both coordinate-free representations of transformations and representations within a particular frame.

- **P, Q, R**: points in an affine space
- **u, v, w**: vectors in an affine space
- **α, β, γ**: scalars
- **p, q, r**: representations of points
  - array of 4 scalars in homogeneous coordinates
- **u, v, w**: representations of vectors
  - array of 4 scalars in homogeneous coordinates

Translation

• Move (translate, displace) a point to a new location

\[ P' = P + d \]

• Displacement determined by a vector \( d \)
  - Three degrees of freedom
  - \( P' = P + d \)
How many ways?

Although we can move a point to a new location in infinite ways, when we move many points there is usually only one way.

translation: every point displaced by same vector
Translation Using Representations

Using the homogeneous coordinate representation in some frame

\[ p = [x \ y \ z \ 1]^T \]
\[ p' = [x' \ y' \ z' \ 1]^T \]
\[ d = [dx \ dy \ dz \ 0]^T \]

Hence \( p' = p + d \) or

\[ x' = x + d_x \]
\[ y' = y + d_y \]
\[ z' = z + d_z \]

Note that this expression is in four dimensions and expresses point = point + vector.
We can also express translation using a 4 x 4 matrix $T$ in homogeneous coordinates $p' = Tp$ where

$$T = T(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This form is better for implementation because all affine transformations can be expressed this way and multiple transformations can be concatenated together.
Translation

\[
\begin{bmatrix}
    x' \\
y' \\
z' \\
1
\end{bmatrix} = \begin{bmatrix}
    1 & 0 & 0 & d_x \\
    0 & 1 & 0 & d_y \\
    0 & 0 & 1 & d_z \\
    0 & 0 & 0 & 1
\end{bmatrix}\begin{bmatrix}
    x \\
y \\
z \\
1
\end{bmatrix}
\]

\[
x' = x + d_x \\
y' = y + d_y \\
z' = z + d_z
\]
Rotation (2D)

Consider rotation about the origin by $\theta$ degrees
- radius stays the same, angle increases by $\theta$

$$x' = r \cos (\phi + \theta)$$
$$y' = r \sin (\phi + \theta)$$

$x' = r \cdot \cos (\theta + \phi) = r \cdot \cos \phi \cdot \cos \theta - r \cdot \sin \phi \cdot \sin \theta$
$$y' = r \cdot \sin (\theta + \phi) = r \cdot \cos \phi \cdot \sin \theta + r \cdot \sin \phi \cdot \cos \theta$$

$x' = x \cos \theta - y \sin \theta$
$$y' = x \sin \theta + y \cos \theta$$
Rotation about the $z$ axis

• Rotation about $z$ axis in three dimensions leaves all points with the same $z$
  
  - Equivalent to rotation in two dimensions in planes of constant $z$
    
    \[
    x' = x \cos \theta - y \sin \theta \\
    y' = x \sin \theta + y \cos \theta \\
    z' = z
    \]

  - or in homogeneous coordinates
    
    \[p' = R_Z(\theta)p\]
Rotation Matrix

\[
R = R_z(\theta) = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 & 0 \\
\sin \theta & \cos \theta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
Rotation Matrix

\[
\begin{bmatrix}
    x' \\
    y' \\
    z' \\
    1
\end{bmatrix}
= \begin{bmatrix}
    \cos \theta & -\sin \theta & 0 & 0 & 0 \\
    \sin \theta & \cos \theta & 0 & 0 & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    x \\
    y \\
    z \\
    1
\end{bmatrix}
\]

\[
x' = x \cos \theta - y \sin \theta
\]
\[
y' = x \sin \theta + y \cos \theta
\]
\[
z' = z
\]
Rotation about $x$ and $y$ axes

- Same argument as for rotation about $z$ axis
  - For rotation about $x$ axis, $x$ is unchanged
  - For rotation about $y$ axis, $y$ is unchanged

\[ R = R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 & 0 \\ 0 & \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ R = R_y(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
Rotation Around an Arbitrary Axis

- Rotate a point \( P \) around axis \( n \) (x,y,z) by angle \( \theta \)

\[
R = \begin{bmatrix}
tx^2 + c & txy + sz & txz - sy & 0 \\
txy - sz & ty^2 + c & tYZ + sx & 0 \\
txz + sy & tYZ - sx & tz^2 + c & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

- \( c = \cos(\theta) \)
- \( s = \sin(\theta) \)
- \( t = (1 - c) \)

Graphics Gems I, p. 466 & 498
Rotation Around an Arbitrary Axis

• Also can be expressed as the Rodrigues Formula

\[ P_{rot} = P \cos(\vartheta) + (\mathbf{n} \times P) \sin(\vartheta) + \mathbf{n}(\mathbf{n} \cdot P)(1 - \cos(\vartheta)) \]
Scaling

Expand or contract along each axis (fixed point of origin)

\[ x' = s_x x \]
\[ y' = s_y y \]
\[ z' = s_z z \]
\[ p' = Sp \]

\[
S = S(s_x, s_y, s_z) = \begin{bmatrix}
  s_x & 0 & 0 & 0 \\
  0 & s_y & 0 & 0 \\
  0 & 0 & s_z & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]
Reflection

corresponds to negative scale factors

\[
\begin{align*}
&s_x = -1 \ s_y = 1 \\
&s_x = -1 \ s_y = -1 \\
&s_x = 1 \ s_y = -1
\end{align*}
\]
Shear

- Helpful to add one more basic transformation
- Equivalent to pulling faces in opposite directions
Consider simple shear along $x$ axis

$$
x' = x + y \cot \theta
\quad y' = y
\quad z' = z
$$

$$
H(\theta) = \begin{bmatrix}
1 & \cot \theta & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 
\end{bmatrix}
$$
Inverses

- Although we could compute inverse matrices by general formulas, we can use simple geometric observations
  - Translation: \( T^{-1}(d_x, d_y, d_z) = T(-d_x, -d_y, -d_z) \)
  - Rotation: \( R^{-1}(\theta) = R(-\theta) \)
    - Holds for any rotation matrix
    - Note that since \( \cos(-\theta) = \cos(\theta) \) and \( \sin(-\theta) = -\sin(\theta) \)
      \[ R^{-1}(\theta) = R^T(\theta) \]
  - Scaling: \( S^{-1}(s_x, s_y, s_z) = S(1/s_x, 1/s_y, 1/s_z) \)
Concatenation

• We can form arbitrary affine transformation matrices by multiplying together rotation, translation, and scaling matrices.

• Because the same transformation is applied to many vertices, the cost of forming a matrix $M = ABCD$ is not significant compared to the cost of computing $Mp$ for many vertices $p$.

• The difficult part is how to form a desired transformation from the specifications in the application.
Order of Transformations

• Note that matrix on the right is the first applied
• Mathematically, the following are equivalent
  \[ p' = (ABC)p = A(B(Cp)) \]  // pre-multiply
• Note many references use column matrices to represent points. In terms of column matrices
  \[ p'^T = p^T C^T B^T A^T \]  // post-multiply
Properties of Transformation Matrices

- Note that matrix multiplication is not commutative
- i.e. in general $M_1M_2 \neq M_2M_1$

- $T$ – reflection around y axis
- $T'$ – rotation in the plane
A rotation by $\theta$ about an arbitrary axis can be decomposed into the concatenation of rotations about the $x$, $y$, and $z$ axes

$$R(\theta) = R_z(\theta_z) R_y(\theta_y) R_x(\theta_x)$$

$\theta_x \theta_y \theta_z$ are called the Euler angles.

Note that rotations do not commute. We can use rotations in another order but with different angles.
Rotation About a Fixed Point other than the Origin

Move fixed point to origin
Rotate
Move fixed point back

\[ M = T(p_f) \, R(\theta) \, T(-p_f) \]
Composition of 2D Transforms

- Rotate about a point $P_1$
  - Translate $P_1$ to origin
  - Rotate
  - Translate back to $P_1$

$$ P' = \mathcal{T} \ast P $$

$$ \mathcal{T} = \begin{bmatrix} 1 & 0 & x_1 \\ 0 & 1 & y_1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -y_1 \\ 0 & 0 & 1 \end{bmatrix} $$

Original house

After translation of $P_1$ to origin

After rotation

After translation to original $P_1$
Composition of 2D Transforms

- Scale object around point $P1$
  - $P1$ to origin
  - Scale
  - Translate back to $P1$
  - Compose into $\mathcal{T}$

$$T(x_1, y_1) \cdot S(S_x, S_y) \cdot T(-x_1, -y_1)$$

$$\begin{bmatrix}
1 & 0 & x_1 \\
0 & 1 & y_1 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
S_x & 0 & 0 \\
0 & S_y & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 & -x_1 \\
0 & 1 & -y_1 \\
0 & 0 & 1
\end{bmatrix}$$

$$\mathcal{T} = \begin{bmatrix}
S_x & 0 & x_1(1 - S_x) \\
0 & S_y & y_1(1 - S_y) \\
0 & 0 & 1
\end{bmatrix}$$

$$P' = \mathcal{T} \cdot P$$
Composition of 2D Transforms

- Scale + rotate object around point $P1$ and move to $P2$
  - $P1$ to origin
  - Scale
  - Rotate
  - Translate to $P2$

$$P' = T \cdot P$$

$$T(x_2, y_2) \cdot R(\theta) \cdot S(s_x, s_y) \cdot T(-x_1, -y_1)$$
Composition of Transformation Matrices

• Be sure to multiple transformations in proper order!

\[ P' = (T \times (R \times (S \times (T \times P)))) \]
\[ P' = ((T \times (R \times (S \times T))) \times P) \]
\[ P' = T \times P \]
Instancing

• In modeling, we often start with a simple object centered at the origin, oriented with the axis, and at a standard size

• We apply an *instance transformation* to its vertices to

  Scale
  Orient
  Locate
Shear

• Helpful to add one more basic transformation
• Equivalent to pulling faces in opposite directions
Shear Matrix

Consider simple shear along $x$ axis

\[ x' = x + y \cot \theta \]
\[ y' = y \]
\[ z' = z \]

\[
H(\theta) = \begin{bmatrix}
1 & \cot \theta & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
WebGL Transformations
Objectives

• Learn how to carry out transformations in WebGL
  - Rotation
  - Translation
  - Scaling

• Introduce MV.js transformations
  - Model-view
  - Projection
Pre 3.1 OpenGL Matrices

• In OpenGL matrices were part of the state
• Multiple types
  - Model-View (GL_MODELVIEW)
  - Projection (GL_PROJECTION)
  - Texture (GL_TEXTURE)
  - Color(GL_COLOR)
• Single set of functions for manipulation
• Select which to manipulated by
  - glMatrixMode(GL_MODELVIEW);
  - glMatrixMode(GL_PROJECTION);
Why Deprecation

• Functions were based on carrying out the operations on the CPU as part of the fixed function pipeline

• Current model-view and projection matrices were automatically applied to all vertices using CPU

• We will use the notion of a current transformation matrix with the understanding that it may be applied in the shaders
Current Transformation Matrix (CTM)

• Conceptually there was a 4 x 4 homogeneous coordinate matrix, the current transformation matrix (CTM) that is part of the state and is applied to all vertices that pass down the pipeline.

• The CTM was defined in the user program and loaded into a transformation unit.

\[
p' = Cp
\]
CTM operations

- The CTM can be altered either by loading a new CTM or by postmultiplication
  - Load an identity matrix: \( \mathbf{C} \leftarrow \mathbf{I} \)
  - Load an arbitrary matrix: \( \mathbf{C} \leftarrow \mathbf{M} \)
  - Load a translation matrix: \( \mathbf{C} \leftarrow \mathbf{T} \)
  - Load a rotation matrix: \( \mathbf{C} \leftarrow \mathbf{R} \)
  - Load a scaling matrix: \( \mathbf{C} \leftarrow \mathbf{S} \)
  - Postmultiply by an arbitrary matrix: \( \mathbf{C} \leftarrow \mathbf{CM} \)
  - Postmultiply by a translation matrix: \( \mathbf{C} \leftarrow \mathbf{CT} \)
  - Postmultiply by a rotation matrix: \( \mathbf{C} \leftarrow \mathbf{CR} \)
  - Postmultiply by a scaling matrix: \( \mathbf{C} \leftarrow \mathbf{CS} \)
Rotation about a Fixed Point

Start with identity matrix: \( C \leftarrow I \)
Move fixed point to origin: \( C \leftarrow CT \)
Rotate: \( C \leftarrow CR \)
Move fixed point back: \( C \leftarrow CT^{-1} \)

Result: \( C = TR \ T^{-1} \) which is backwards.

This result is a consequence of doing postmultiplications. Let’s try again.
Reversing the Order

We want \( C = T^{-1} R T \)
so we must do the operations in the following order

\[
\begin{align*}
C & \leftarrow I \\
C & \leftarrow CT^{-1} \\
C & \leftarrow CR \\
C & \leftarrow CT
\end{align*}
\]

Note that the last operation specified is the first executed in the program

Recall \( p' = ABCp = A(B(Cp)) \quad // \text{pre-multiply} \)
CTM in WebGL

• OpenGL had a model-view and a projection matrix in the pipeline which were concatenated together to form the CTM
• We will emulate this process
Using the ModelView Matrix

• In WebGL, the model-view matrix is used to
  - Position the camera
    • Can be done by rotations and translations but is
      often easier to use the lookAt function in MV.js
  - Build models of objects
• The projection matrix is used to define the
  view volume and to select a camera lens
• Although these matrices are no longer part of
  the OpenGL state, it is usually a good strategy
  to create them in our own applications

\[ q = P \times MV \times p \]

Angel and Shreiner: Interactive Computer Graphics 7E © Addison-Wesley 2015
MV.js

Defines basic types

• vec2, vec3, vec4
  - 2, 3 & 4 element arrays

• mat2, mat3, mat4
  - 2x2, 3x3 & 4x4 matrices

• Operators
  - equal, add, subtract, mult, transpose, dot, cross, length, normalize, flatten, det, inverse

• Generators
  - Rotate, RotateX, RotateY, RotateZ, Translate, Scale
  - Ortho, Ortho2D, Frustum, Perspective, LookAt
Create an identity matrix:

```javascript
var m = mat4();
```

Multiply on right by rotation matrix of `theta` in degrees where $(vx, vy, vz)$ define axis of rotation

```javascript
var r = rotate(theta, vx, vy, vz)
    m = mult(m, r);
```

Also have `rotateX`, `rotateY`, `rotateZ`

Do same with translation and scaling:

```javascript
var s = scale(sx, sy, sz)
    var t = translate(dx, dy, dz);
    m = mult(s, t);
```
Example

• Rotation about z axis by 30 degrees with a fixed point of (1.0, 2.0, 3.0)

var m = mult(translate(1.0, 2.0, 3.0),
  rotate(30.0, 0.0, 0.0, 1.0));
m = mult(m, translate(-1.0, -2.0, -3.0));

• Remember that last matrix specified in the program is the first applied
Arbitrary Matrices

- Can load and multiply by matrices defined in the application program.
- Matrices are stored as one dimensional array of 16 elements by MV.js but can be treated as 4 x 4 matrices in row major order.
- OpenGL wants column major data.
- `gl.uniformMatrix4f` has a parameter for automatic transpose, but it must be set to false.
- `flatten` function converts to column major order which is required by WebGL functions.
Matrix Stacks

• In many situations we want to save transformation matrices for use later
  - Traversing hierarchical data structures (Chapter 9)
• Pre 3.1 OpenGL maintained stacks for each type of matrix
• Easy to create the same functionality in JS
  - push and pop are part of Array object
    var stack = [ ]
    stack.push(modelViewMatrix);
    modelViewMatrix = stack.pop();
Applying Transformations
Using Transformations

- Example: Begin with a cube rotating
- Use mouse or button listener to change direction of rotation
- Start with a program that draws a cube in a standard way
  - Centered at origin
  - Sides aligned with axes
  - Will discuss modeling in next lecture
Where do we apply transformation?

• Same issue as with rotating square
  - in application to vertices
  - in vertex shader: send MV matrix
  - in vertex shader: send angles

• Choice between second and third unclear

• Do we do trigonometry once in CPU or for every vertex in shader
  - GPUs have trig functions hardwired in silicon
Rotation Event Listeners

document.getElementById( "xButton" ).onclick = function () {
  axis = xAxis; 
};
document.getElementById( "yButton" ).onclick = function () {
  axis = yAxis; 
};
document.getElementById( "zButton" ).onclick = function () {
  axis = zAxis; 
};

function render(){
  gl.clear( gl.COLOR_BUFFER_BIT | gl.DEPTH_BUFFER_BIT);
  theta[axis] += 2.0;
  gl.uniform3fv(thetaLoc, theta);
  gl.drawArrays( gl.TRIANGLES, 0, NumVertices );
  requestAnimFrame( render );
}
in vec4 aPosition;      // Streaming vertices and colors
in vec4 aColor;         // from the application
out vec4 vColor;        // Sending color to fragment shader
uniform vec3 theta;     // Getting theta from application

void main() {
    vec3 angles = radians( theta );
    vec3 c = cos( angles );
    vec3 s = sin( angles );
    // Remember: these matrices are column-major!
    mat4 rx = mat4( 1.0, 0.0, 0.0, 0.0,
                    0.0, c.x, s.x, 0.0,
                    0.0, -s.x, c.x, 0.0,
                    0.0, 0.0, 0.0, 1.0 );

Rotation Shader (cont)

mat4 ry = mat4(
    c.y, 0.0, -s.y, 0.0,
    0.0, 1.0, 0.0, 0.0,
    s.y, 0.0, c.y, 0.0,
    0.0, 0.0, 0.0, 1.0);

mat4 rz = mat4(
    c.z, s.z, 0.0, 0.0,
    -s.z, c.z, 0.0, 0.0,
    0.0, 0.0, 1.0, 0.0,
    0.0, 0.0, 0.0, 1.0);

vColor = aColor;

gl_Position = rz * ry * rx * aPosition;
Smooth Rotation

• From a practical standpoint, we often want to use transformations to move and reorient an object smoothly
  - Problem: find a sequence of model-view matrices $M_0, M_1, \ldots, M_n$ so that when they are applied successively to one or more objects we see a smooth transition

• For orientating an object, we can use the fact that every rotation corresponds to part of a great circle on a sphere
  - Find the axis of rotation and angle
  - Virtual trackball (see text)
• Consider the two approaches
  - For a sequence of rotation matrices $R_0, R_1, \ldots, R_n$, find the Euler angles for each and use $R_i = R_{iz} R_{iy} R_{ix}$
    • Not very efficient
  - Use the final positions to determine the axis and angle of rotation, then increment only the angle
• Quaternions can be more efficient than either
Improved Rotations

• Euler Angles have problems
  - How to interpolate keyframes?
  - Angles aren’t independent
    - Interpolation can create Gimble Lock, i.e. loss of a degree of freedom when axes align

• Solution: Quaternions!
Quaternions

- Extension of imaginary numbers from two to three dimensions
- Requires one real and three imaginary components $i$, $j$, $k$

$$q = q_0 + q_1 i + q_2 j + q_3 k$$

- Quaternions can express rotations on sphere smoothly and efficiently. Process:
  - Model-view matrix $\rightarrow$ quaternion
  - Carry out operations with quaternions
  - Quaternion $\rightarrow$ Model-view matrix
Quaternions

Matrices are not the only (or best) way of representing rotations. For one thing, they are redundant (9 numbers instead of 3) and, for another, they are difficult to interpolate.

An alternative representation was developed by Hamilton in the early 19th century (and forgotten until relatively recently). The quaternion is a 4-tuple of reals with the operations of addition and multiplication defined. Just as complex numbers allow us to multiply and divide two-dimensional vectors, quaternions enable us to multiply and divide four dimensional vectors.

\[ q = q_0 + q_1 i + q_2 j + q_3 k \]

\[ i^2 = j^2 = k^2 = -1 \quad ij = k, jk = i, ki = j \]

A quaternion can also be interpreted as having a scalar part and a vector part. This will give us a more convenient notation.

\[ q = (s, \vec{a}) \quad \text{pure quaternion : } \quad p = (0, \vec{x}) \]

Quaternion addition is just the usual vector addition, the quaternion product is defined as:

\[ q_1 q_2 = (s_1 s_2 - (\vec{a}_1 \cdot \vec{a}_2), s_1 \vec{a}_2 + s_2 \vec{a}_1 + \vec{a}_1 \times \vec{a}_2) \]
Quatertion Facts

conjugate: \[ q^* = (s, -\vec{a}) \]

magnitude: \[ |q| = \sqrt{qq^*} = \sqrt{s^2 + \vec{a} \cdot \vec{a}} \]

unit quaternion: \[ |q| = 1 \]

inverse: \[ q^{-1} = \frac{1}{|q|^2} q^* \]

\[ q^{-1} = q^*, \text{ for unit quaternions} \]

It turns out that we will be able to represent rotations with a unit quaternion. Before looking at why this is so, there are a few important properties to keep in mind:

- The unit quaternions form a three-dimensional sphere in the 4-dimensional space of quaternions.
- Any quaternion can be interpreted as a rotation simply by normalizing it (dividing it by its length).
- Both \( q \) and \( -q \) represent the same rotation (corresponding to angles of \( q \) and \( 2p - q \))
Rotation by Quaternion

\[ R_q(p) = q p q^{-1} \]
\[ p = (0, \bar{x}) \]

\[ q = (\cos(\theta/2), \sin(\theta/2)\bar{a}), \quad \text{where} \ |\bar{a}| = 1 \]

\[ R_q(p) = (0, \quad (s^2 - \bar{a} \cdot \bar{a})\bar{x} \]
\[ + \quad 2\bar{a}(\bar{a} \cdot \bar{x}) \]
\[ + \quad 2s(\bar{a} \times \bar{x}) \]

\[ R_q(p) = (0, \quad (\cos^2(\theta/2) - \sin^2(\theta/2))\bar{x} \]
\[ + \quad (2\sin^2(\theta/2))\bar{a}(\bar{a} \cdot \bar{x}) \]
\[ + \quad (2\cos(\theta/2)\sin(\theta/2))(\bar{a} \times \bar{x}) \]

\[ R_q(p) = (0, \quad (\cos \theta)\bar{x} \]
\[ + \quad (1 - \cos \theta)\bar{a}(\bar{a} \cdot \bar{x}) \]
\[ + \quad (\sin \theta)(\bar{a} \times \bar{x}) \]

Recognize this? It is the Rodrigues formula!
Quaternion Composition

Since a quaternion basically stores the axis vector and angle of rotation, it is not surprising that we can write the components of a rotation matrix given the quaternion components.

\[ q = (\cos(\theta/2), \sin(\theta/2)\vec{a}) = (w, (x, y, z)) \]

\[
R_q = \begin{pmatrix}
1 - 2y^2 - 2z^2 & 2xy - 2wz & 2xz + 2wy & 0 \\
2xy + 2wz & 1 - 2x^2 - 2z^2 & 2yz - 2wx & 0 \\
2xz - 2wy & 2yz + 2wx & 1 - 2x^2 - 2y^2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Crucially, the composition of two rotations given by quaternions is simply their quaternion product.

\[ R_q'(R_q(p)) = R_{q''}(p) \quad \text{where } q'' = q'q \]

- Note that the product of two unit quaternions is another unit quaternion.
- Note that quaternion multiplication, like matrix multiplication, is not commutative.
From rotation matrix to quaternion

Given $R = (r_{ij})$, solve expression on previous page for quaternion elements $q_i$

Linear combinations of diagonal elements seem to solve the problem:

$$q_0^2 = \frac{1}{4} (1 + r_{11} + r_{22} + r_{33})$$

$$q_1^2 = \frac{1}{4} (1 + r_{11} - r_{22} - r_{33})$$

$$q_2^2 = \frac{1}{4} (1 - r_{11} + r_{22} - r_{33})$$

$$q_3^2 = \frac{1}{4} (1 - r_{11} - r_{22} + r_{33})$$

so take four square roots and you’re done? You have to figure the signs out. There is a better way ...
Quaternion Interpolation

One of the main motivations for using quaternions in Graphics is the ease with which we can define interpolation between two orientations. Think, for example, about moving a camera smoothly between two views.

A & B are quaternions

$$\cos \Omega = A \cdot B$$

$$C(t) = \text{slerp}(A, B, t) = \frac{A \sin(\Omega(1-t))}{\sin \Omega} + \frac{B \sin(\Omega t)}{\sin \Omega}$$

slerp – Spherical linear interpolation

Need to take equals steps on the sphere
What about interpolating multiple keyframes?

- Shoemake suggests using Bezier curves on the sphere
- Offers a variation of the De Casteljau algorithm using slerp and quaternion control points