CS 536
Computer Graphics

Circle Drawing
Week 6, Lecture 10

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Outline

• Scan conversion of circles
• Clipping circles
• Scan conversion of ellipses
Scan Conversion of Circles

- Generalization of the line algorithm
- Assumptions:
  - circle at (0,0)
  - Fill 1/8 of the circle, then use symmetry

Using the 8-way symmetry of a circle:

Not using the 8-way symmetry of a circle
Scan Conversion of Circles

- Implicit representation of the circle function:

\[ F(x, y) = x^2 + y^2 - R^2 = 0. \]

- Note: \( F(x, y) < 0 \) for points inside the circle, and \( F(x, y) > 0 \) for points outside the circle.
Scan Conversion of Circles

- Assume we finished pixel $(x_p, y_p)$
- What pixel to draw next? (going clockwise)
- Note: the slope of the circular arc is between 0 and $-1$
  - Hence, choice is between: $E$ and $SE$
- Idea:
  If the circle passes above the midpoint $M$, then we go to $E$
  next, otherwise we go to $SE$
Scan Conversion of Circles

• We need a decision variable D:

\[
D = F(M) = F(x_p + 1, y_p - \frac{1}{2})
\]

\[
= (x_p + 1)^2 + (y_p - \frac{1}{2})^2 - R^2.
\]

• If \( D < 0 \) then \( M \) is below the arc, hence the \( E \) pixel is closer to the line.
• If \( D \geq 0 \) then \( M \) is above the arc, hence the \( SE \) pixel is closer to the line.
Case I: When $E$ is next

- What increment for computing a new $D$?
- Next midpoint is: \((x_p + 2, y_p - (1/2))\)

\[
D_{\text{new}} = F(x_p + 2, y_p - \frac{1}{2})
\]

\[
= (x_p + 2)^2 + (y_p - \frac{1}{2})^2 - R^2
\]

\[
= (x_p^2 + 4x_p + 4) + (y_p - \frac{1}{2})^2 - R^2
\]

\[
= (x_p^2 + 2x_p + 1) + (2x_p + 3) + (y_p - \frac{1}{2})^2 - R^2
\]

\[
= (x_p + 1)^2 + (2x_p + 3) + (y_p - \frac{1}{2})^2 - R^2
\]

\[
= D + (2x_p + 3).
\]

- Hence, increment by: \((2x_p + 3)\)
Case II: When SE is next

• What increment for computing a new $D$?
• Next midpoint is: $(x_p + 2, y_p - 1 - (1/2))$

\[
D_{new} = F(x_p + 2, y_p - \frac{3}{2})
\]
\[
= (x_p + 2)^2 + (y_p - \frac{3}{2})^2 - R^2
\]
\[
= (x_p^2 + 4x_p + 4) + (y_p^2 - 3y_p + \frac{9}{4}) - R^2
\]
\[
= (x_p^2 + 2x_p + 1) + (2x_p + 3) + (y_p^2 - y_p + \frac{1}{4}) + (-2y_p + \frac{8}{4}) - R^2
\]
\[
= (x_p + 1)^2 + (y_p - \frac{1}{2})^2 - R^2 + (2x_p + 3) + (-2y_p + 2)
\]
\[
= D + (2x_p - 2y_p + 5)
\]

• Hence, increment by: $(2x_p - 2y_p + 5)$
Scan Conversion of Circles

- How to compute the *initial* value of $D$:
- We start with $x = 0$ and $y = R$, so the first midpoint is at $x = 1$, $y = R - 1/2$:

\[
D_{\text{init}} = F(1, R - \frac{1}{2})
\]

\[
= 1 + (R - \frac{1}{2})^2 - R^2
\]

\[
= 1 + R^2 - R + \frac{1}{4} - R^2
\]

\[
= \frac{5}{4} - R.
\]
Scan Conversion of Circles

• Converting this to an integer algorithm:
  – Need only know if $D$ is positive or negative
  – $D$ & $R$ are integers
  – Note $D$ is incremented by an integer value
  – Therefore $D + 1/4$ is positive only when $D$ is positive; it is safe to drop the $1/4$

• Hence: set the initial $D$ to $1 - R$
  (subtracting $1/4$)
Circle Scan Conversion Algorithm

• Given radius $R$ and center $(0, 0)$
  – First point $\rightarrow (0, R)$
• Initial decision parameter $D = 1 - R$
• While $x \leq y$
  – If $(D < 0)$
    • $x++; D += 2x + 3$;
  – else
    • $x++; y--; D += 2(x - y) + 5$
  – WritePoints(x, y)
WritePoints(x,y)

- Writes pixels to the seven other octants
Clipping Circles

• **Accept/Reject test**
  - Does bounding box of the circle intersect with clipping box?
• **If yes, condition pixel write on clipping box inside/outside test**
Properties of Ellipses

- “Elongated circle”
- For all points on ellipse, sum of distances to foci is constant
  \[ d_1 + d_2 = \text{const} \]
- If \( F_1 = (x_1, y_1) \) and \( F_2 = (x_2, y_2) \) then
  \[ \sqrt{(x - x_1)^2 + (y - y_1)^2} + \sqrt{(x - x_2)^2 + (y - y_2)^2} = \text{const} \]
Properties of Ellipses

- Equation simplified if ellipse axis parallel to coordinate axis

\[
\left(\frac{x - x_c}{r_x}\right)^2 + \left(\frac{y - y_c}{r_y}\right)^2 = 1
\]

- Parametric form

\[
x = x_c + r_x \cos \theta \\
y = y_c + r_y \sin \theta
\]
Symmetry Considerations

- 4-way symmetry
- Unit steps in $x$ until reach region boundary
- Switch to unit steps in $y$

$$f(x, y) = r_y^2 x^2 + r_x^2 y^2 - r_x^2 r_y^2$$

$$\frac{dy}{dx} = -\frac{r_y^2 x}{r_x^2 y}$$

$$\frac{dy}{dx} = -1$$

$$r_y^2 x = r_x^2 y$$

- Step in $x$ while
  $$r_y^2 x < r_x^2 y$$
- Switch to steps in $y$ when
  $$r_y^2 x \geq r_x^2 y$$
**Midpoint Algorithm (initializing)**

- Similar to circles
- The initial value for region 1
  
  \[ D_{init1} = f(1, r_y - \frac{1}{2}) \]
  
  \[ = r_y^2 - r_x^2 r_y + \frac{1}{4} r_x^2 \]
- The initial value for region 2
  
  \[ D_{init2} = f(x_p + \frac{1}{2}, y_p - 1) \]
  
  \[ = r_y^2 (x_p + \frac{1}{2})^2 + r_x^2 (y_p - 1)^2 - r_x^2 r_y^2 \]
- We have initial values, now we need the increments
Making a Decision

- Computing the decision variable

\[ D = f(x_p + 1, y_p - \frac{1}{2}) \]
\[ = r_y^2 (x_p + 1)^2 + r_x^2 (y_p - \frac{1}{2})^2 - r_x^2 r_y^2 \]

- If \( D < 0 \) then \( M \) is below the arc, hence the \( E \) pixel is closer to the line.
- If \( D \geq 0 \) then \( M \) is above the arc, hence the \( SE \) pixel is closer to the line.
Computing the Increment

E case
\[ D_{\text{new}} = f(x_p + 2, y_p - \frac{1}{2}) \]
\[ = r_y^2 (x_p + 2)^2 + r_x^2 (y_p - \frac{1}{2})^2 - r_x r_y^2 \]
\[ = D_{\text{old}} + r_y^2 (2x_p + 3) \]

SE case
\[ D_{\text{new}} = f(x_p + 2, y_p - \frac{3}{2}) \]
\[ = r_y^2 (x_p + 2)^2 + r_x^2 (y_p - \frac{3}{2})^2 - r_x r_y^2 \]
\[ = D_{\text{old}} + r_y^2 (2x_p + 3) + r_x^2 (-2y_p + 2) \]

increment = \[ \begin{cases} 
  r_y^2 (2x_p + 3) & D_{\text{old}} < 0 \\
  r_y^2 (2x_p + 3) + r_x^2 (-2y_p + 2) & D_{\text{old}} \geq 0 
\end{cases} \]
Computing the Increment in 2\textsuperscript{nd} Region

- Decision variable in 2\textsuperscript{nd} region

\[ D = f \left( x_p + \frac{1}{2}, y_p - 1 \right) \]
\[ = r_y^2 \left( x_p + \frac{1}{2} \right)^2 + r_x^2 (y_p - 1)^2 - r_x^2 r_y^2 \]

- If \( D < 0 \) then \( M \) is left of the arc, hence the \( SE \) pixel is closer to the line.
- If \( D \geq 0 \) then \( M \) is right of the arc, hence the \( S \) pixel is closer to the line.
Computing the Increment in 2\textsuperscript{nd} Region

SE case

\[ D_{\text{new}} = f(x_p + \frac{3}{2}, y_p - 2) \]
\[ = r_y^2 (x_p + \frac{3}{2})^2 + r_x^2 (y_p - 2)^2 - r_x r_y^2 \]
\[ = D_{\text{old}} + r_x^2 (-2y_p + 3) + r_y^2 (2x_p + 2) \]

S case

\[ D_{\text{new}} = f(x_p + \frac{1}{2}, y_p - 2) \]
\[ = r_y^2 (x_p + \frac{1}{2})^2 + r_x^2 (y_p - 2)^2 - r_x r_y^2 \]
\[ = D_{\text{old}} + r_x^2 (-2y_p + 3) \]

increment = \[ \begin{cases} 
  r_x^2 (-2y_p + 3) + r_y^2 (2x_p + 2) & D_{\text{old}} < 0 \\
  r_y^2 (-2y_p + 3) & D_{\text{old}} \geq 0 
\end{cases} \]
Midpoint Algorithm for Ellipses

Region 1

Set first point to (0, r_y)
Set the Decision variable to
\[ D_{init1} = r_y^2 - r_x^2 r_y + \frac{1}{4} r_x^2 \]
Loop \((x = x + 1)\)
    If \( D < 0 \) then pick \( E \) and
    \[ D = r_y^2 (2x_p + 3) \]
    If \( D \geq 0 \) then pick \( SE \) and
    \[ D = r_y^2 (2x_p + 3) + r_x^2 (-2y_p + 2) \]
Until \( 2r_y^2 x_k \geq 2r_x^2 y_k \)

Region 2

Set first point to the last computed
Set the Decision variable to
\[ D_{init2} = r_y^2 (x_p + \frac{1}{2})^2 + r_x^2 (y_p - 1)^2 - r_x^2 r_y \]
Loop \((y = y - 1)\)
    If \( D < 0 \) then pick \( SE \) and
    \[ D = r_y^2 (2x_p + 2) + r_x^2 (-2y_p + 3) \]
    If \( D \geq 0 \) then pick \( S \) and
    \[ D = r_x^2 (-2y_p + 3) \]
Until \( y < 0 \)
Use symmetry to complete the ellipse