Geometry

CS 537 Interactive Computer Graphics
Prof. David E. Breen
Department of Computer Science

Objectives

- Introduce the elements of geometry
  - Scalars
  - Vectors
  - Points
- Develop mathematical operations among them in a coordinate-free manner
- Define basic primitives
  - Line segments
  - Polygons

Basic Elements

- Geometry is the study of the relationships among objects in an n-dimensional space
  - In computer graphics, we are interested in objects that exist in three dimensions
- Want a minimum set of primitives from which we can build more sophisticated objects
- We will need three basic elements
  - Scalars
  - Vectors
  - Points

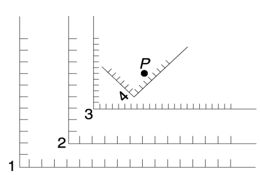
Coordinate-Free Geometry

- When we learned simple geometry, most of us started with a Cartesian approach
  - Points were at locations in space **p**=(x,y,z)
  - We derived results by algebraic manipulations involving these coordinates
- This approach was nonphysical
  - Physically, points exist regardless of the location of an arbitrary coordinate system
  - Most geometric results are independent of the coordinate system
  - Example Euclidean geometry: two triangles are identical if two corresponding sides and the angle between them are identical

Transformations to Change Coordinate Systems

- 4 coordinate systems
  1 point *P*

$$M_{1\leftarrow 2} = T(4,2)$$
$$M_{2\leftarrow 3} = T(2,3) \bullet S(0.5,0.5)$$
$$M_{3\leftarrow 4} = T(6.7,1.8) \bullet R(45^\circ)$$



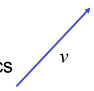
$$M_{i\leftarrow k} = M_{i\leftarrow j} \cdot M_{j\leftarrow k}$$

Scalars

- Need three basic elements in geometry
  - Scalars, Vectors, Points
- Scalars can be defined as members of sets which can be combined by two operations (addition and multiplication) obeying some fundamental axioms (associativity, commutivity, inverses)
- Examples include the real and complex number systems under the ordinary rules with which we are familiar
- Scalars alone have no geometric properties

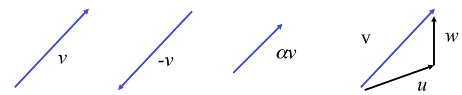
Vectors

- Physical definition: a vector is a quantity with two attributes
  - Direction
  - Magnitude
- Examples include
  - Force
  - Velocity
  - Directed line segments
    - Most important example for graphics
    - Can map to other types



Vector Operations

- Every vector has an inverse
  - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
- There is a zero vector
  - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
  - Use head-to-tail axiom



Linear Vector Spaces

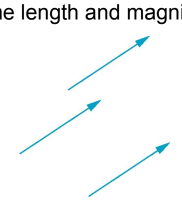
- Mathematical system for manipulating vectors
- Operations
  - Scalar-vector multiplication $u=\alpha v$
  - Vector-vector addition: $w=u+v$
- Expressions such as

  $v=u+2w-3r$

Make sense in a vector space

Vectors Lack Position

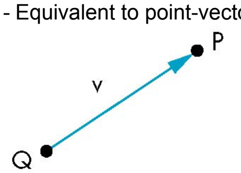
- These vectors are identical
  - Same length and magnitude



- Vectors spaces insufficient for geometry
  - Need points

Points

- Location in space
- Operations allowed between points and vectors
  - Point-point subtraction yields a vector
  - Equivalent to point-vector addition



$v=P-Q$

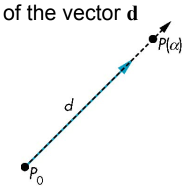
$P=v+Q$

Affine Spaces

- Point + a vector space
- Operations
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition
  - Scalar-scalar operations
- For any point define
  - $1 \bullet P = P$
  - $0 \bullet P = \mathbf{0}$ (zero vector)

Lines

- Consider all points of the form
  - $P(\alpha)=P_0 + \alpha\,\mathbf{d}$
  - Set of all points that pass through P_0 in the direction of the vector $\mathbf{d}$



Parametric Form

- This form is known as the parametric form of the line
  - More robust and general than other forms
  - Extends to curves and surfaces
- Two-dimensional forms
  - Explicit: $y = mx + h$
  - Implicit: $ax + by + c = 0$
  - Parametric:

$$x(\alpha) = \alpha x_0 + (1-\alpha)x_1$$
$$y(\alpha) = \alpha y_0 + (1-\alpha)y_1$$

Rays and Line Segments

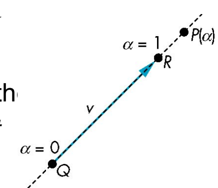
- If $\alpha >= 0$, then $P(\alpha)$ is the *ray* leaving P_0 in the direction $\mathbf{d}$

If we use two points to define v, then

$P(\alpha) = Q + \alpha(R-Q) = Q+\alpha v$

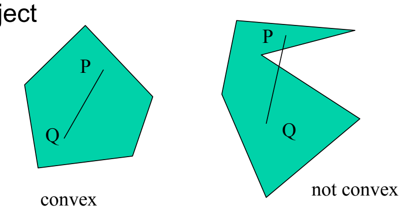
$= \alpha R + (1-\alpha)Q$

For $0 <= \alpha < =1$ we get all the points on the *line segment* joining R and Q



Convexity

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object



Affine Sums

- Consider the "sum"

  $P=\alpha_1 P_1+\alpha_2 P_2+\ldots..+\alpha_n P_n$

If

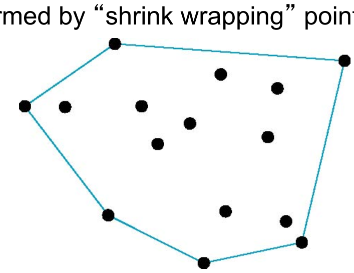
  $\alpha_1+\alpha_2+\ldots..\alpha_n=1$

in which case we have the *affine sum* of the points $P_1,P_2,\ldots..P_n$

- If, in addition, $\alpha_i>=0$, we have the *convex hull* of $P_1,P_2,\ldots..P_n$

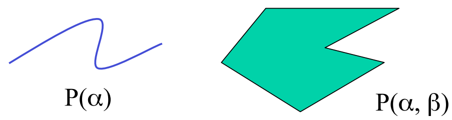
Convex Hull

- Smallest convex object containing $P_1,P_2,\ldots..P_n$
- Formed by "shrink wrapping" points



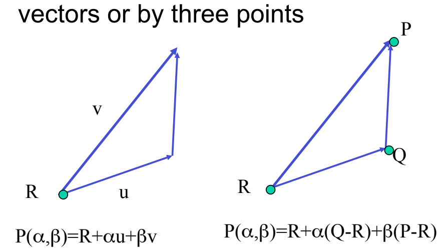
Curves and Surfaces

- Curves are one parameter entities of the form $P(\alpha)$ where the function is nonlinear
- Surfaces are formed from two-parameter functions $P(\alpha, \beta)$
  - Linear functions give planes and polygons

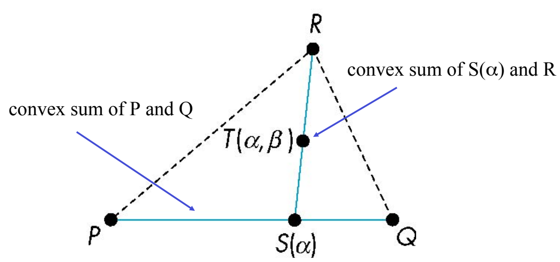


Planes

- A plane can be defined by a point and two vectors or by three points



Triangles



for $0<=\alpha,\beta<=1$, we get all points in triangle

Barycentric Coordinates

Triangle is convex so any point inside can be represented as an affine sum

$P(\alpha_1, \alpha_2, \alpha_3)=\alpha_1 P+\alpha_2 Q+\alpha_3 R$

where

$\alpha_1 +\alpha_2 +\alpha_3 = 1$

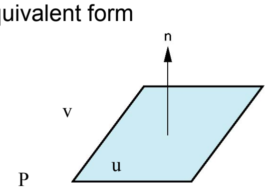
$\alpha_i>=0$

The representation is called the **barycentric coordinate** representation of P

Normals

- Every plane has a vector **n** normal (perpendicular, orthogonal) to it
- From point-two vector form $P(\alpha,\beta)=R+\alpha u+\beta v$, we know we can use the cross product to find $n = u \times v$ and the equivalent form

  $(P(\alpha)-P) \cdot n=0$



Representation

Objectives

- Introduce concepts such as dimension and basis
- Introduce coordinate systems for representing vectors spaces and frames for representing affine spaces
- Discuss change of frames and bases
- Introduce homogeneous coordinates

Linear Independence

- A set of vectors $v_1, v_2, \ldots, v_n$ is *linearly independent* if

  $\alpha_1 v_1 + \alpha_2 v_2 + .. \alpha_n v_n = 0$ iff $\alpha_1 = \alpha_2 = \ldots = 0$
- If a set of vectors is linearly independent, we cannot represent one in terms of the others
- If a set of vectors is linearly dependent, at least one can be written in terms of the others

Dimension

- In a vector space, the maximum number of linearly independent vectors is fixed and is called the *dimension* of the space
- In an *n*-dimensional space, any set of n linearly independent vectors form a *basis* for the space
- Given a basis $v_1, v_2, \ldots, v_n$, any vector v can be written as

  $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$

where the $\{\alpha_i\}$ are unique

Representation

- Until now we have been able to work with geometric entities without using any frame of reference, such as a coordinate system
- Need a frame of reference to relate points and objects to our physical world.
  - For example, where is a point? Can't answer without a reference system
  - World coordinates
  - Camera coordinates

Coordinate Systems

- Consider a basis $v_1, v_2, \ldots, v_n$
- A vector is written $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$
- The list of scalars $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is the *representation* of v with respect to the given basis
- We can write the representation as a row or column array of scalars

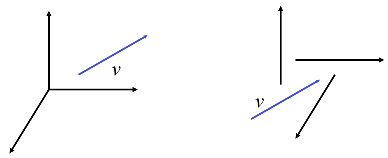
$$\mathbf{a} = [\alpha_1 \ \alpha_2 \ \ldots \ \alpha_n]^T = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_n \end{bmatrix}$$

Example

- $v = 2v_1 + 3v_2 - 4v_3$
- $\mathbf{a} = [2, 3, -4]^T$
- Note that this representation is with respect to a particular basis
- For example, in OpenGL we start by representing vectors using the object basis but later the system needs a representation in terms of the camera or eye basis

Coordinate Systems

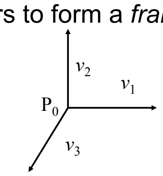
- Which is correct?



- Both are because vectors have no fixed location

Frames

- A coordinate system is insufficient to represent points
- If we work in an affine space we can add a single point, the *origin*, to the basis vectors to form a *frame*



Representation in a Frame

- Frame determined by (P_0, v_1, v_2, v_3)
- Within this frame, every vector can be written as

  $v=\alpha_1 v_1+ \alpha_2 v_2 +\ldots.+\alpha_n v_n$
- Every point can be written as

  $P = P_0 + \beta_1 v_1+ \beta_2 v_2 +\ldots.+\beta_n v_n$

Confusing Points and Vectors

Consider the point and the vector

$P = P_0 + \beta_1 v_1+ \beta_2 v_2 +\ldots.+\beta_n v_n$

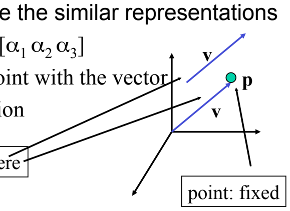
$v=\alpha_1 v_1+ \alpha_2 v_2 +\ldots.+\alpha_n v_n$

They appear to have the similar representations

$\mathbf{p}=[\beta_1\ \beta_2\ \beta_3]$ $\qquad$ $\mathbf{v}=[\alpha_1\ \alpha_2\ \alpha_3]$

which confuses the point with the vector

A vector has no position



A Single Representation

If we define $0\bullet P = \mathbf{0}$ and $1\bullet P = P$ then we can write

$v=\alpha_1 v_1+ \alpha_2 v_2 +\alpha_3 v_3 = [\alpha_1\ \alpha_2\ \alpha_3\ 0]\ [v_1\ v_2\ v_3\ P_0]^T$

$P = P_0 + \beta_1 v_1+ \beta_2 v_2 +\beta_3 v_3= [\beta_1\ \beta_2\ \beta_3\ 1]\ [v_1\ v_2\ v_3\ P_0]^T$

Thus we obtain the four-dimensional *homogeneous coordinate* representation

$\mathbf{v} = [\alpha_1\ \alpha_2\ \alpha_3\ 0]^T$

$\mathbf{p} = [\beta_1\ \beta_2\ \beta_3\ 1]^T$

Homogeneous Coordinates

The homogeneous coordinates form for a three dimensional point [x y z] is given as

$\mathbf{p} =[x'\ y'\ z'\ w]^T =[wx\ wy\ wz\ w]^T$

We return to a three dimensional point (for $w\neq 0$) by

$x \leftarrow x'/w$

$y \leftarrow y'/w$

$z \leftarrow z'/w$

If $w=0$, the representation is that of a vector

Note that homogeneous coordinates replaces points in three dimensions by lines through the origin in four dimensions

For w=1, the representation of a point is [x y z 1]

Homogeneous Coordinates and Computer Graphics

- Homogeneous coordinates are key to all computer graphics systems
  - All standard transformations (rotation, translation, scaling) can be implemented with matrix multiplications using 4 x 4 matrices
  - Hardware pipeline works with 4 dimensional representations
  - For orthographic viewing, we can maintain w=0 for vectors and w=1 for points
  - For perspective we need a *perspective division*

Change of Coordinate Systems

- Consider two representations of the same vector with respect to two different bases. The representations are

$$\mathbf{a}=[\alpha_1\ \alpha_2\ \alpha_3]$$
$$\mathbf{b}=[\beta_1\ \beta_2\ \beta_3]$$

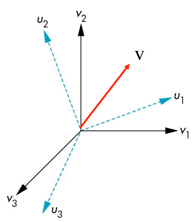
where

$$v=\alpha_1 v_1+\alpha_2 v_2+\alpha_3 v_3=[\alpha_1\ \alpha_2\ \alpha_3]\ [v_1\ v_2\ v_3]^T$$
$$=\beta_1 u_1+\beta_2 u_2+\beta_3 u_3=[\beta_1\ \beta_2\ \beta_3]\ [u_1\ u_2\ u_3]^T$$

Representing second basis in terms of first

Each of the basis vectors, u1,u2, u3, are vectors that can be represented in terms of the first basis

$$u_1=\gamma_{11}v_1+\gamma_{12}v_2+\gamma_{13}v_3$$
$$u_2=\gamma_{21}v_1+\gamma_{22}v_2+\gamma_{23}v_3$$
$$u_3=\gamma_{31}v_1+\gamma_{32}v_2+\gamma_{33}v_3$$



Matrix Form

The coefficients define a 3 x 3 matrix

$$\mathbf{M}=\begin{bmatrix}\gamma_{11} & \gamma_{12} & \gamma_{13}\\ \gamma_{21} & \gamma_{22} & \gamma_{23}\\ \gamma_{31} & \gamma_{32} & \gamma_{33}\end{bmatrix}$$

and the bases can be related by

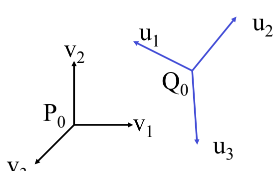
$$\mathbf{a}=\mathbf{M}^T\mathbf{b}$$

see text for numerical examples

Change of Frames

- We can apply a similar process in homogeneous coordinates to the representations of both points and vectors

Consider two frames:
(P_0, v_1, v_2, v_3)
(Q_0, u_1, u_2, u_3)



- Any point or vector can be represented in either frame
- We can represent Q_0, u_1, u_2, u_3 in terms of P_0, v_1, v_2, v_3

Representing One Frame in Terms of the Other

Extending what we did with change of bases

$$u_1=\gamma_{11}v_1+\gamma_{12}v_2+\gamma_{13}v_3$$
$$u_2=\gamma_{21}v_1+\gamma_{22}v_2+\gamma_{23}v_3$$
$$u_3=\gamma_{31}v_1+\gamma_{32}v_2+\gamma_{33}v_3$$
$$Q_0=\gamma_{41}v_1+\gamma_{42}v_2+\gamma_{43}v_3+\gamma_{44}P_0$$

defining a 4 x 4 matrix

$$\mathbf{M}=\begin{bmatrix}\gamma_{11} & \gamma_{12} & \gamma_{13} & 0\\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0\\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0\\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1\end{bmatrix}$$

Working with Representations

Within the two frames any point or vector has a representation of the same form

$\mathbf{a}=[\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4]$ in the first frame
$\mathbf{b}=[\beta_1\ \beta_2\ \beta_3\ \beta_4]$ in the second frame

where $\alpha_4 = \beta_4 = 1$ for points and $\alpha_4 = \beta_4 = 0$ for vectors and

$$\mathbf{a}=\mathbf{M}^T\mathbf{b}$$

The matrix $\mathbf{M}$ is 4 x 4 and specifies an affine transformation in homogeneous coordinates

The World and Camera Frames

- When we work with representations, we work with n-tuples or arrays of scalars
- Changes in frame are then defined by 4 x 4 matrices
- In OpenGL, the base frame that we start with is the world frame
- Eventually we represent entities in the camera frame by changing the world representation using the model-view matrix
- Initially these frames are the same ($\mathbf{M}=\mathbf{I}$)

Moving the Camera

If objects are on both sides of z=0, we must move camera frame

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$