



Geometry

CS 537 Interactive Computer Graphics

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Objectives

- Introduce the elements of geometry
 - Scalars
 - Vectors
 - Points
- Develop mathematical operations among them in a coordinate-free manner
- Define basic primitives
 - Line segments
 - Polygons



Basic Elements

- Geometry is the study of the relationships among objects in an n-dimensional space
 - In computer graphics, we are interested in objects that exist in three dimensions
- Want a minimum set of primitives from which we can build more sophisticated objects
- We will need three basic elements
 - Scalars
 - Vectors
 - Points



Coordinate-Free Geometry

- When we learned simple geometry, most of us started with a Cartesian approach
 - Points were at locations in space $\mathbf{p}=(x,y,z)$
 - We derived results by algebraic manipulations involving these coordinates
- This approach was nonphysical
 - Physically, points exist regardless of the location of an arbitrary coordinate system
 - Most geometric results are independent of the coordinate system
 - Example Euclidean geometry: two triangles are identical if two corresponding sides and the angle between them are identical



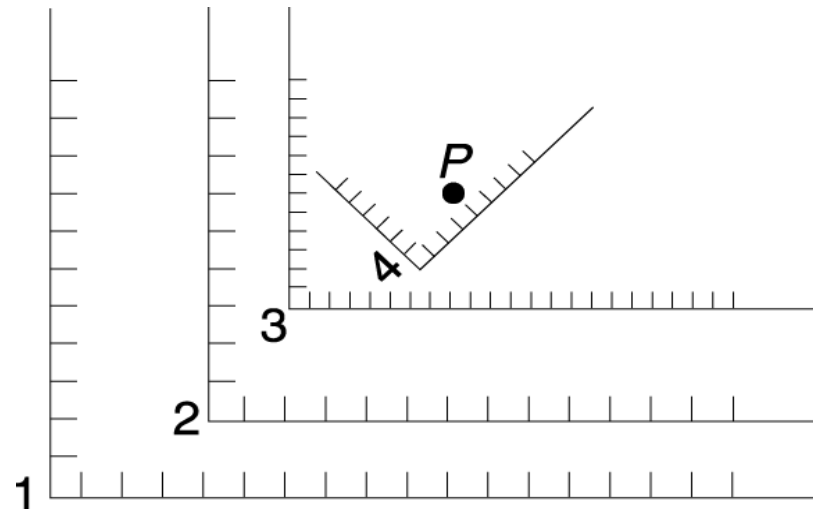
Transformations to Change Coordinate Systems

- 4 coordinate systems
- 1 point P

$$M_{1 \leftarrow 2} = T(4, 2)$$

$$M_{2 \leftarrow 3} = T(2, 3) \cdot S(0.5, 0.5)$$

$$M_{3 \leftarrow 4} = T(6.7, 1.8) \cdot R(45^\circ)$$



$$M_{i \leftarrow k} = M_{i \leftarrow j} \cdot M_{j \leftarrow k}$$



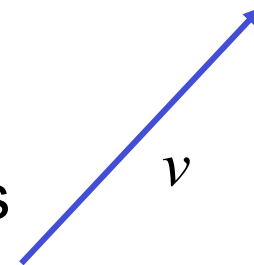
Scalars

- Need three basic elements in geometry
 - Scalars, Vectors, Points
- Scalars can be defined as members of sets which can be combined by two operations (addition and multiplication) obeying some fundamental axioms (associativity, commutivity, inverses)
- Examples include the real and complex number systems under the ordinary rules with which we are familiar
- Scalars alone have no geometric properties



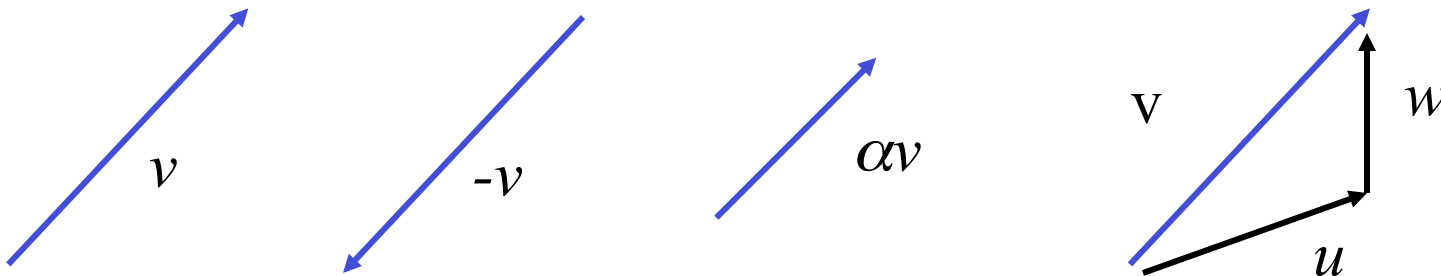
Vectors

- Physical definition: a vector is a quantity with two attributes
 - Direction
 - Magnitude
- Examples include
 - Force
 - Velocity
 - Directed line segments
 - Most important example for graphics
 - Can map to other types



Vector Operations

- Every vector has an inverse
 - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
- There is a zero vector
 - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
 - Use head-to-tail axiom





Linear Vector Spaces

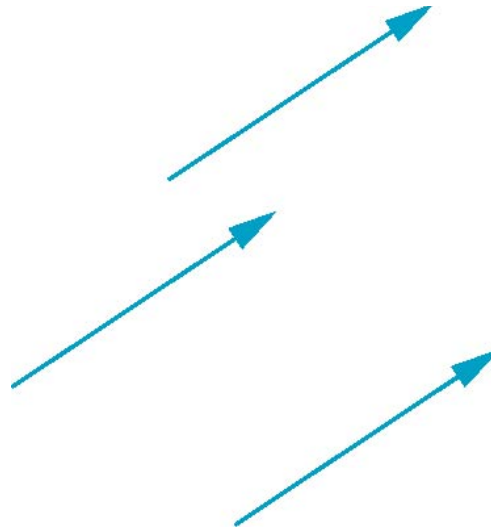
- Mathematical system for manipulating vectors
- Operations
 - Scalar-vector multiplication $u = \alpha v$
 - Vector-vector addition: $w = u + v$
- Expressions such as
$$v = u + 2w - 3r$$

Make sense in a vector space



Vectors Lack Position

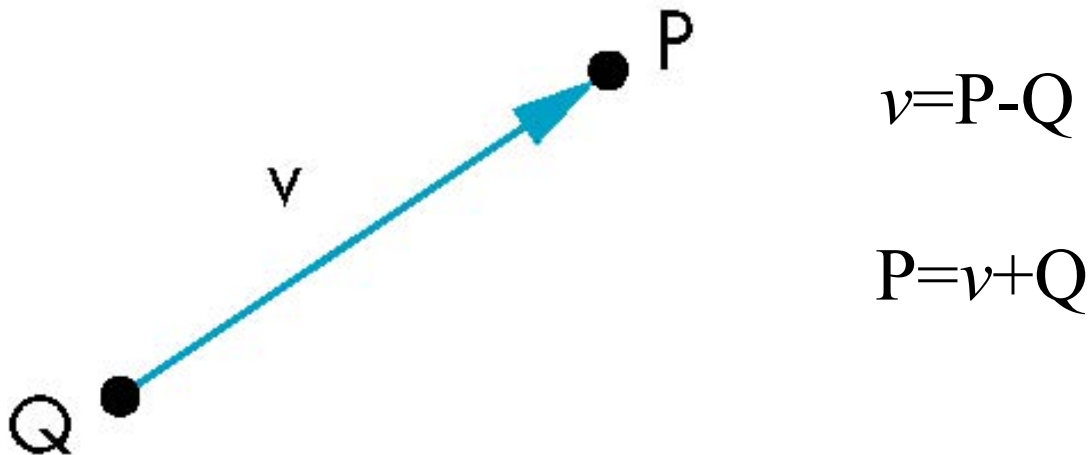
- These vectors are identical
 - Same length and magnitude



- Vectors spaces insufficient for geometry
 - Need points

Points

- Location in space
- Operations allowed between points and vectors
 - Point-point subtraction yields a vector
 - Equivalent to point-vector addition



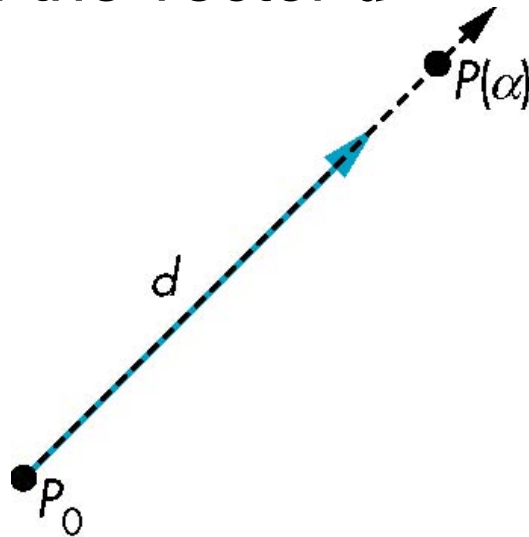


Affine Spaces

- Point + a vector space
- Operations
 - Vector-vector addition
 - Scalar-vector multiplication
 - Point-vector addition
 - Scalar-scalar operations
- For any point define
 - $1 \cdot P = P$
 - $0 \cdot P = \mathbf{0}$ (zero vector)

Lines

- Consider all points of the form
 - $P(\alpha) = P_0 + \alpha \mathbf{d}$
 - Set of all points that pass through P_0 in the direction of the vector \mathbf{d}





Parametric Form

- This form is known as the parametric form of the line
 - More robust and general than other forms
 - Extends to curves and surfaces
- Two-dimensional forms
 - Explicit: $y = mx + h$
 - Implicit: $ax + by + c = 0$
 - Parametric:

$$x(\alpha) = \alpha x_0 + (1-\alpha)x_1$$

$$y(\alpha) = \alpha y_0 + (1-\alpha)y_1$$

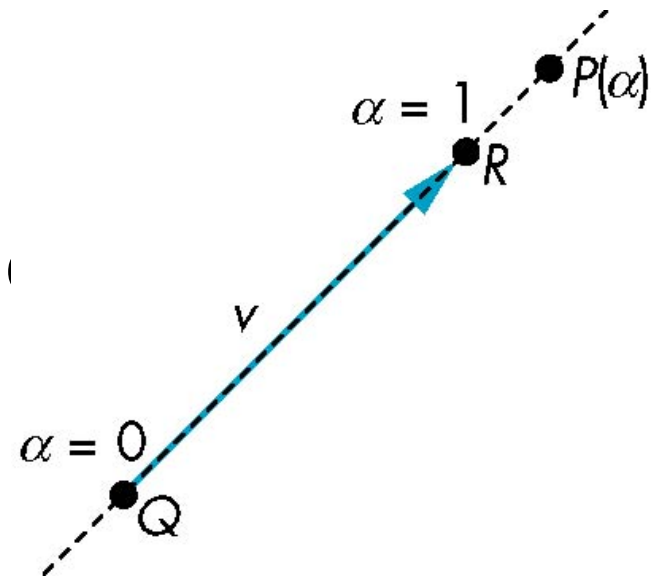
Rays and Line Segments

- If $\alpha \geq 0$, then $P(\alpha)$ is the *ray* leaving P_0 in the direction \mathbf{d}

If we use two points to define \mathbf{v} , then

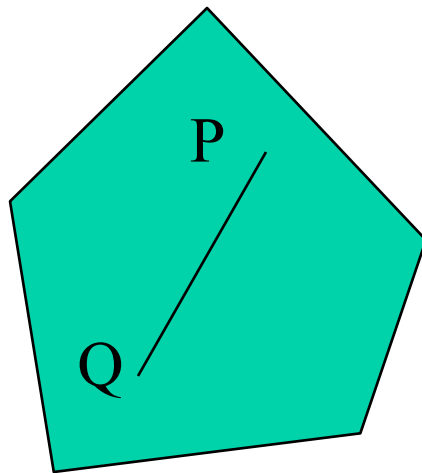
$$\begin{aligned} P(\alpha) &= Q + \alpha (R - Q) = Q + \alpha \mathbf{v} \\ &= \alpha R + (1 - \alpha)Q \end{aligned}$$

For $0 \leq \alpha \leq 1$ we get all the points on the *line segment* joining R and Q

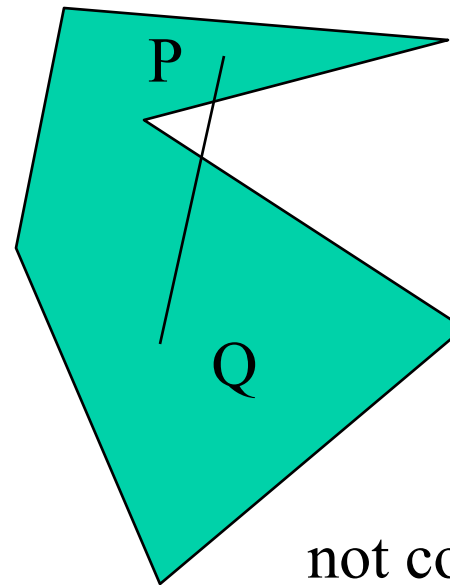


Convexity

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object



convex



not convex



Affine Sums

- Consider the “sum”

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_n P_n$$

If

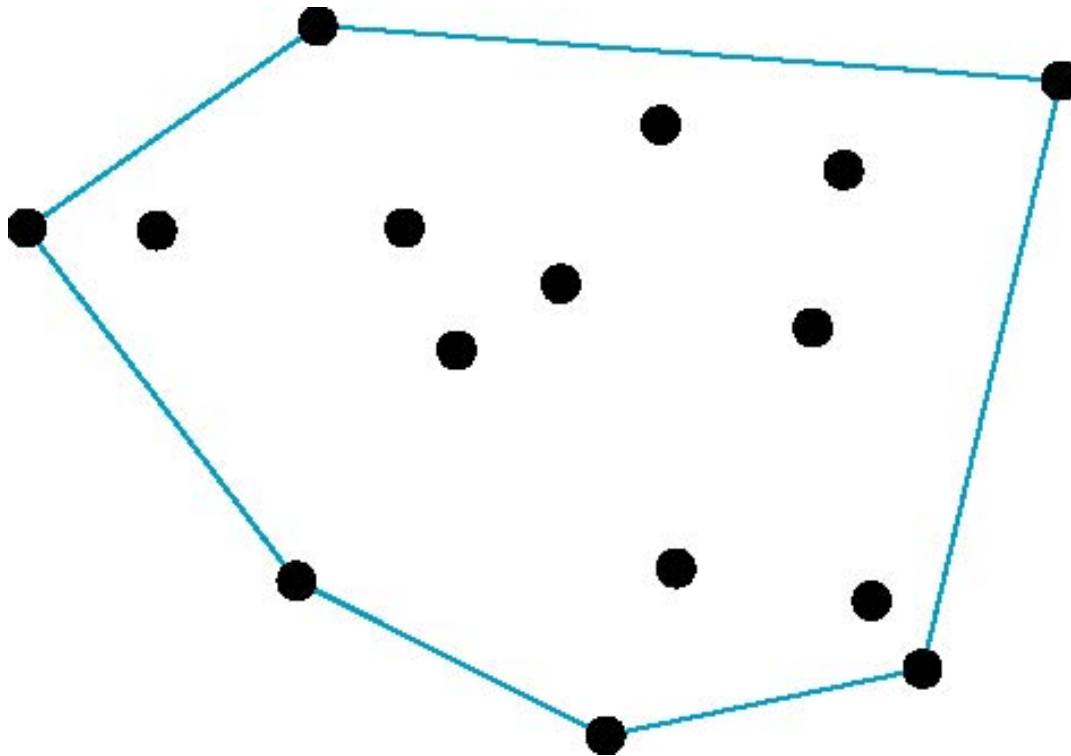
$$\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$$

in which case we have the *affine sum* of the points P_1, P_2, \dots, P_n

- If, in addition, $\alpha_i \geq 0$, we have the *convex hull* of P_1, P_2, \dots, P_n

Convex Hull

- Smallest convex object containing P_1, P_2, \dots, P_n
- Formed by “shrink wrapping” points

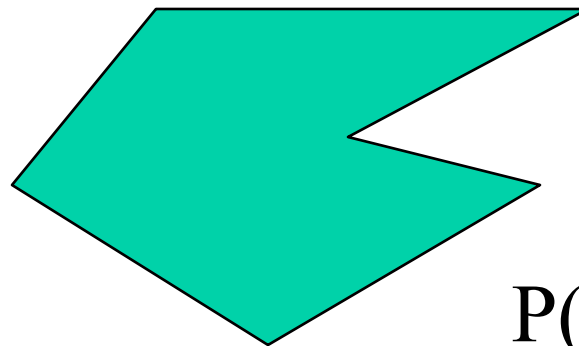


Curves and Surfaces

- Curves are one parameter entities of the form $P(\alpha)$ where the function is nonlinear
- Surfaces are formed from two-parameter functions $P(\alpha, \beta)$
 - Linear functions give planes and polygons



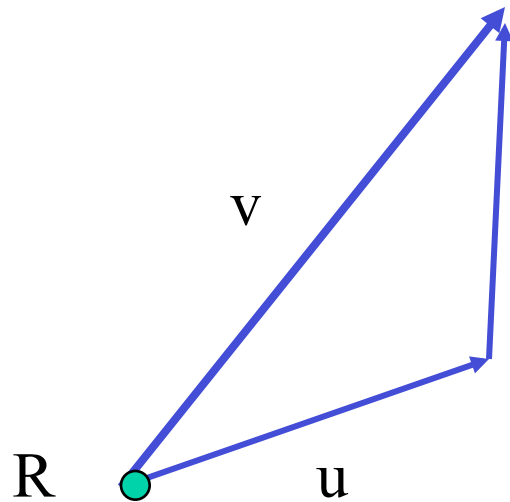
$P(\alpha)$



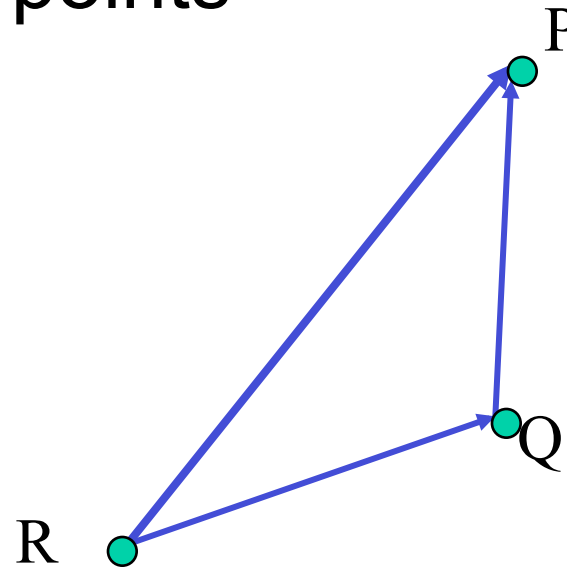
$P(\alpha, \beta)$

Planes

- A plane can be defined by a point and two vectors or by three points

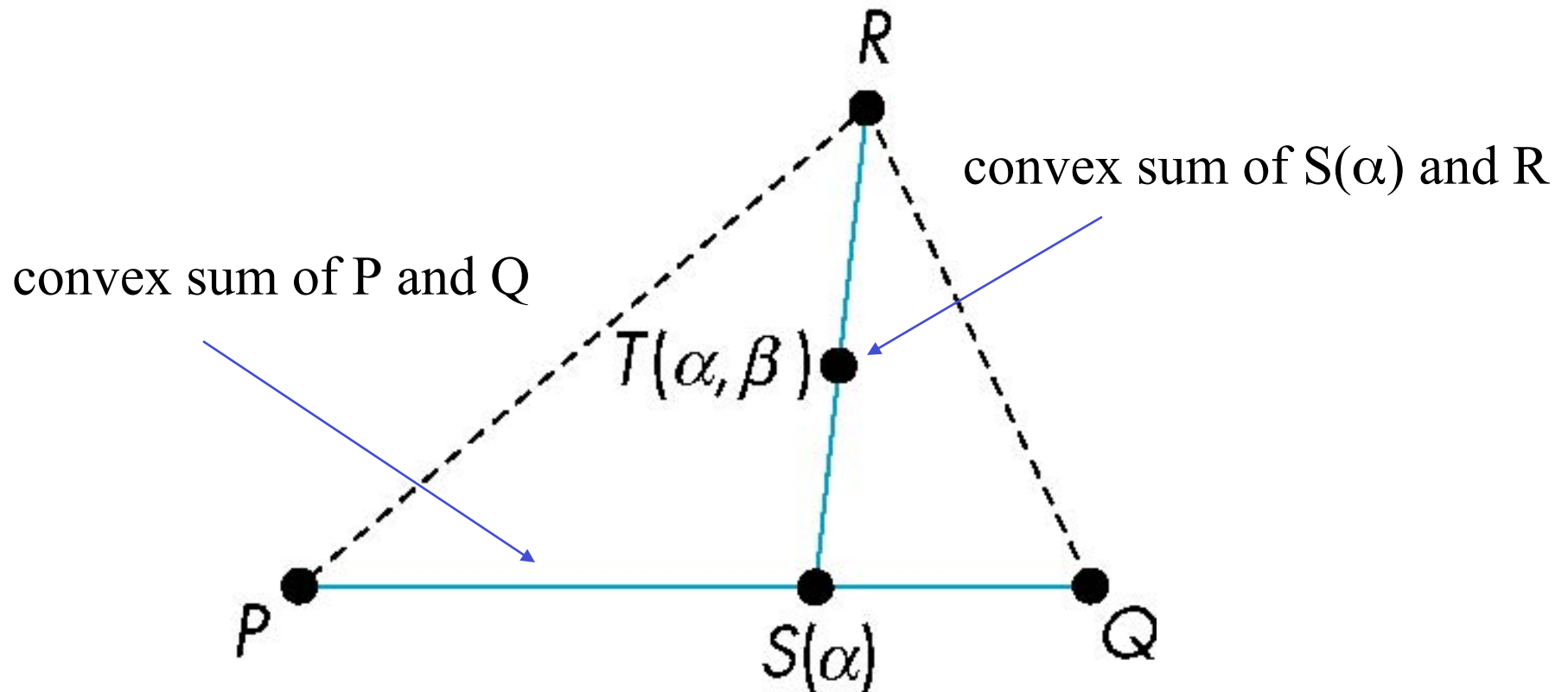


$$P(\alpha, \beta) = R + \alpha u + \beta v$$



$$P(\alpha, \beta) = R + \alpha(Q - R) + \beta(P - R)$$

Triangles



for $0 \leq \alpha, \beta \leq 1$, we get all points in triangle



Barycentric Coordinates

Triangle is convex so any point inside can be represented as an affine sum

$$P(\alpha_1, \alpha_2, \alpha_3) = \alpha_1 P + \alpha_2 Q + \alpha_3 R$$

where

$$\alpha_1 + \alpha_2 + \alpha_3 = 1$$

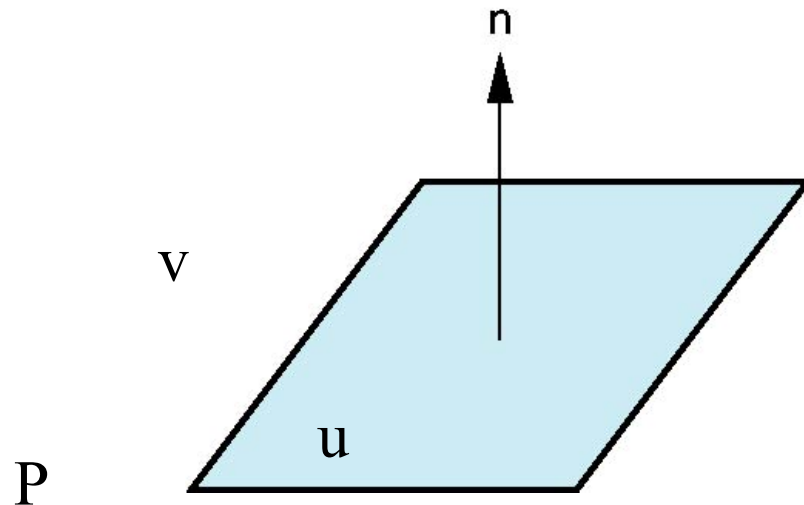
$$\alpha_i \geq 0$$

The representation is called the **barycentric coordinate** representation of P

Normals

- Every plane has a vector \mathbf{n} normal (perpendicular, orthogonal) to it
- From point-two vector form $P(\alpha, \beta) = R + \alpha\mathbf{u} + \beta\mathbf{v}$, we know we can use the cross product to find $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ and the equivalent form

$$(\mathbf{P}(\alpha) - \mathbf{P}) \cdot \mathbf{n} = 0$$





Representation



Objectives

- Introduce concepts such as dimension and basis
- Introduce coordinate systems for representing vectors spaces and frames for representing affine spaces
- Discuss change of frames and bases
- Introduce homogeneous coordinates



Linear Independence

- A set of vectors v_1, v_2, \dots, v_n is *linearly independent* if

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \text{ iff } \alpha_1 = \alpha_2 = \dots = 0$$

- If a set of vectors is linearly independent, we cannot represent one in terms of the others
- If a set of vectors is linearly dependent, at least one can be written in terms of the others



Dimension

- In a vector space, the maximum number of linearly independent vectors is fixed and is called the *dimension* of the space
- In an n -dimensional space, any set of n linearly independent vectors form a *basis* for the space
- Given a basis v_1, v_2, \dots, v_n , any vector v can be written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

where the $\{\alpha_i\}$ are unique



Representation

- Until now we have been able to work with geometric entities without using any frame of reference, such as a coordinate system
- Need a frame of reference to relate points and objects to our physical world.
 - For example, where is a point? Can't answer without a reference system
 - World coordinates
 - Camera coordinates



Coordinate Systems

- Consider a basis v_1, v_2, \dots, v_n
- A vector is written $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$
- The list of scalars $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is the *representation* of v with respect to the given basis
- We can write the representation as a row or column array of scalars

$$\mathbf{a} = [\alpha_1 \quad \alpha_2 \quad \dots \quad \alpha_n]^T = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \alpha_n \end{bmatrix}$$

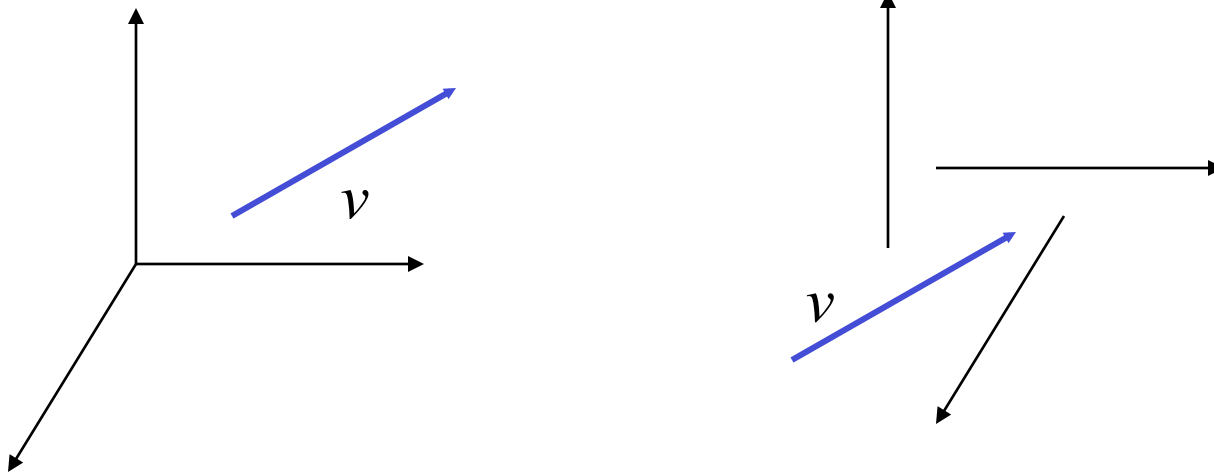


Example

- $\mathbf{v} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 4\mathbf{v}_3$
- $\mathbf{a} = [2, 3, -4]^T$
- Note that this representation is with respect to a particular basis
- For example, in OpenGL we start by representing vectors using the object basis but later the system needs a representation in terms of the camera or eye basis

Coordinate Systems

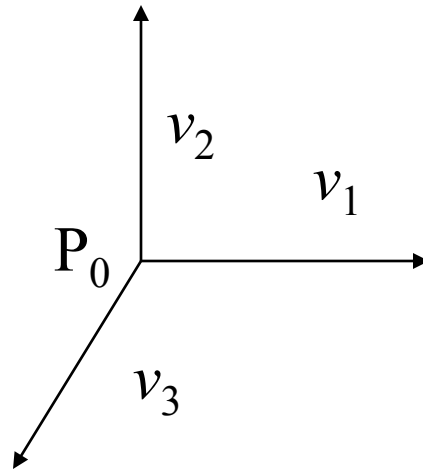
- Which is correct?



- Both are because vectors have no fixed location

Frames

- A coordinate system is insufficient to represent points
- If we work in an affine space we can add a single point, the *origin*, to the basis vectors to form a *frame*





Representation in a Frame

- Frame determined by (P_0, v_1, v_2, v_3)
- Within this frame, every vector can be written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

- Every point can be written as

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$$



Confusing Points and Vectors

Consider the point and the vector

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$$

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

They appear to have the similar representations

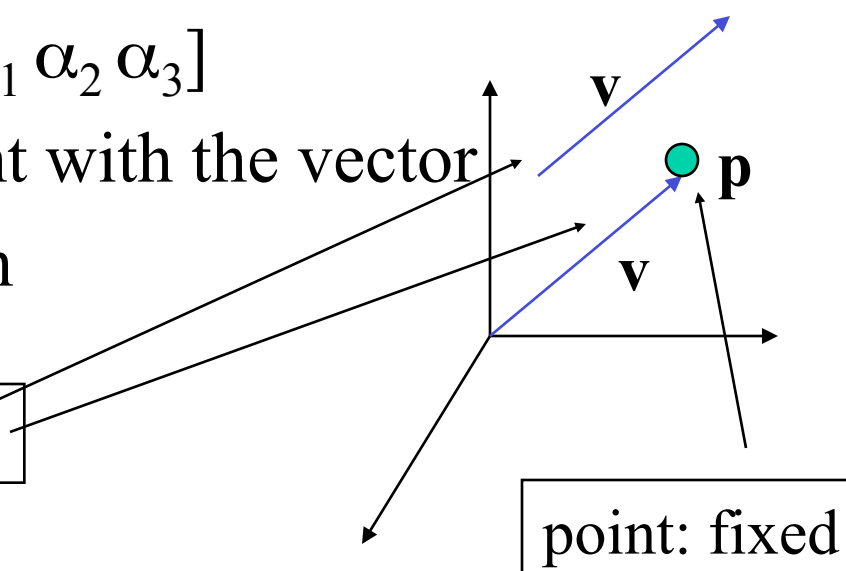
$$\mathbf{p} = [\beta_1 \ \beta_2 \ \beta_3] \quad \mathbf{v} = [\alpha_1 \ \alpha_2 \ \alpha_3]$$

which confuses the point with the vector

A vector has no position

Vector can be placed anywhere

point: fixed





A Single Representation

If we define $0 \cdot P = \mathbf{0}$ and $1 \cdot P = P$ then we can write

$$\mathbf{v} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = [\alpha_1 \ \alpha_2 \ \alpha_3 \ 0] [v_1 \ v_2 \ v_3 \ P_0]^T$$

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = [\beta_1 \ \beta_2 \ \beta_3 \ 1] [v_1 \ v_2 \ v_3 \ P_0]^T$$

Thus we obtain the four-dimensional

homogeneous coordinate representation

$$\mathbf{v} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ 0]^T$$

$$\mathbf{p} = [\beta_1 \ \beta_2 \ \beta_3 \ 1]^T$$



Homogeneous Coordinates

The homogeneous coordinates form for a three dimensional point $[x \ y \ z]$ is given as

$$\mathbf{p} = [x' \ y' \ z' \ w]^T = [wx \ wy \ wz \ w]^T$$

We return to a three dimensional point (for $w \neq 0$) by

$$x \leftarrow x' / w$$

$$y \leftarrow y' / w$$

$$z \leftarrow z' / w$$

If $w=0$, the representation is that of a vector

Note that homogeneous coordinates replaces points in three dimensions by lines through the origin in four dimensions

For $w=1$, the representation of a point is $[x \ y \ z \ 1]$



Homogeneous Coordinates and Computer Graphics

- Homogeneous coordinates are key to all computer graphics systems
 - All standard transformations (rotation, translation, scaling) can be implemented with matrix multiplications using 4 x 4 matrices
 - Hardware pipeline works with 4 dimensional representations
 - For orthographic viewing, we can maintain $w=0$ for vectors and $w=1$ for points
 - For perspective we need a *perspective division*



Change of Coordinate Systems

- Consider two representations of the same vector with respect to two different bases. The representations are

$$\mathbf{a} = [\alpha_1 \ \alpha_2 \ \alpha_3]$$

$$\mathbf{b} = [\beta_1 \ \beta_2 \ \beta_3]$$

where

$$\begin{aligned} \mathbf{v} &= \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = [\alpha_1 \ \alpha_2 \ \alpha_3] [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]^T \\ &= \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \beta_3 \mathbf{u}_3 = [\beta_1 \ \beta_2 \ \beta_3] [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3]^T \end{aligned}$$



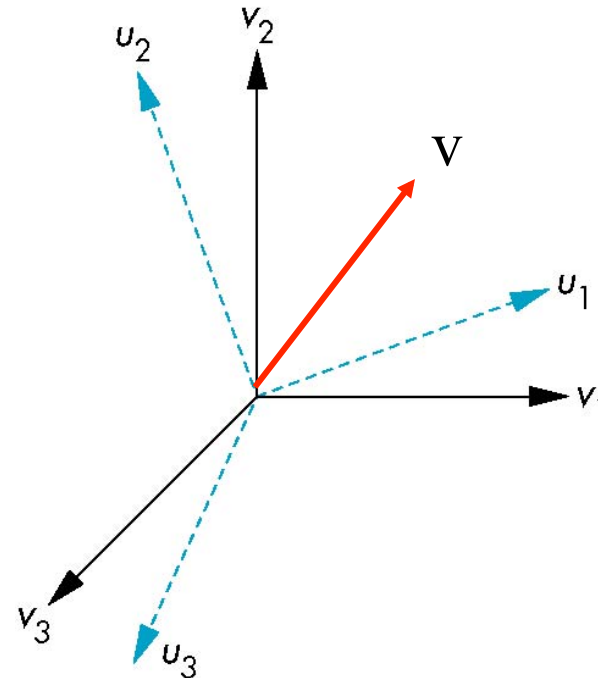
Representing second basis in terms of first

Each of the basis vectors, u_1, u_2, u_3 , are vectors that can be represented in terms of the first basis

$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$$

$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$$

$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$$





Matrix Form

The coefficients define a 3 x 3 matrix

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

and the bases can be related by

$$\mathbf{a} = \mathbf{M}^T \mathbf{b}$$

see text for numerical examples

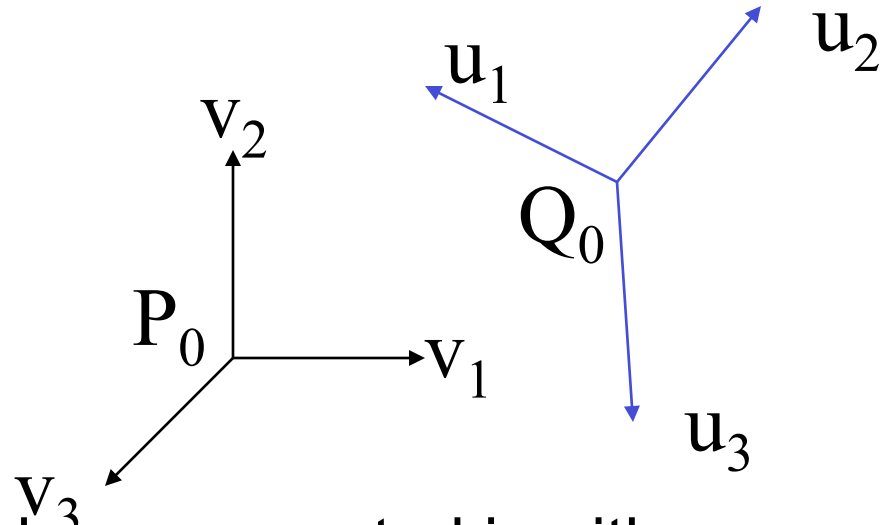
Change of Frames

- We can apply a similar process in homogeneous coordinates to the representations of both points and vectors

Consider two frames:

(P_0, v_1, v_2, v_3)

(Q_0, u_1, u_2, u_3)



- Any point or vector can be represented in either frame
- We can represent Q_0, u_1, u_2, u_3 in terms of P_0, v_1, v_2, v_3



Representing One Frame in Terms of the Other

Extending what we did with change of bases

$$\mathbf{u}_1 = \gamma_{11}\mathbf{v}_1 + \gamma_{12}\mathbf{v}_2 + \gamma_{13}\mathbf{v}_3$$

$$\mathbf{u}_2 = \gamma_{21}\mathbf{v}_1 + \gamma_{22}\mathbf{v}_2 + \gamma_{23}\mathbf{v}_3$$

$$\mathbf{u}_3 = \gamma_{31}\mathbf{v}_1 + \gamma_{32}\mathbf{v}_2 + \gamma_{33}\mathbf{v}_3$$

$$\mathbf{Q}_0 = \gamma_{41}\mathbf{v}_1 + \gamma_{42}\mathbf{v}_2 + \gamma_{43}\mathbf{v}_3 + \gamma_{44}\mathbf{P}_0$$

defining a 4 x 4 matrix

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$



Working with Representations

Within the two frames any point or vector has a representation of the same form

$\mathbf{a} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4]$ in the first frame

$\mathbf{b} = [\beta_1 \ \beta_2 \ \beta_3 \ \beta_4]$ in the second frame

where $\alpha_4 = \beta_4 = 1$ for points and $\alpha_4 = \beta_4 = 0$ for vectors and

$$\mathbf{a} = \mathbf{M}^T \mathbf{b}$$

The matrix \mathbf{M} is 4 x 4 and specifies an affine transformation in homogeneous coordinates



The World and Camera Frames

- When we work with representations, we work with n-tuples or arrays of scalars
- Changes in frame are then defined by 4 x 4 matrices
- In OpenGL, the base frame that we start with is the world frame
- Eventually we represent entities in the camera frame by changing the world representation using the model-view matrix
- Initially these frames are the same ($\mathbf{M}=\mathbf{I}$)

Moving the Camera

If objects are on both sides of $z=0$, we must move camera frame

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

