CS 430
Computer Graphics

Circle Drawing and Clipping

Week 3, Lecture 6

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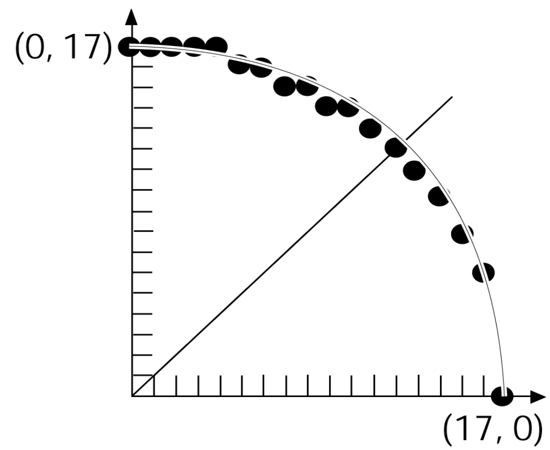
Outline

- Scan conversion of circles
- Clipping circles
- Scan conversion of ellipses

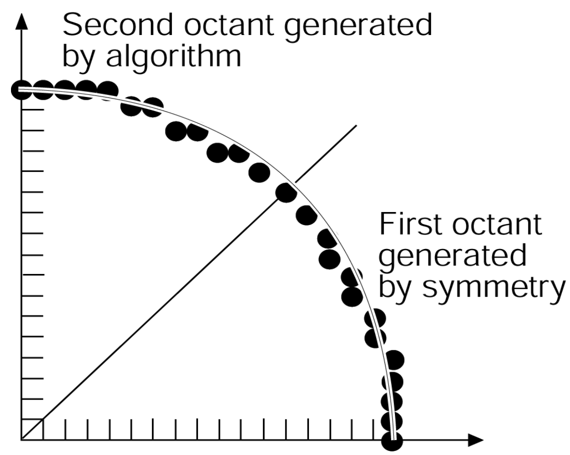
Scan Conversion of Circles

- Generalization of the line algorithm
- Assumptions:
  - circle at (0,0)
  - Fill 1/8 of the circle, then use symmetry



Not using the 8-way symmetry of a circle

Using the 8-way symmetry of a circle:



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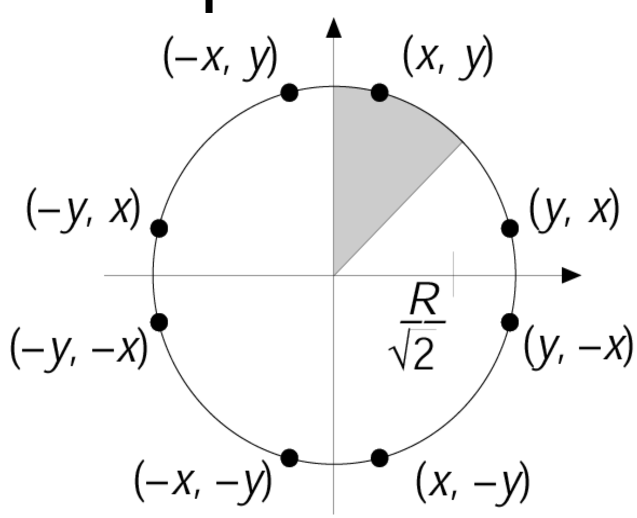
Scan Conversion of Circles

- Implicit representation of the circle function:

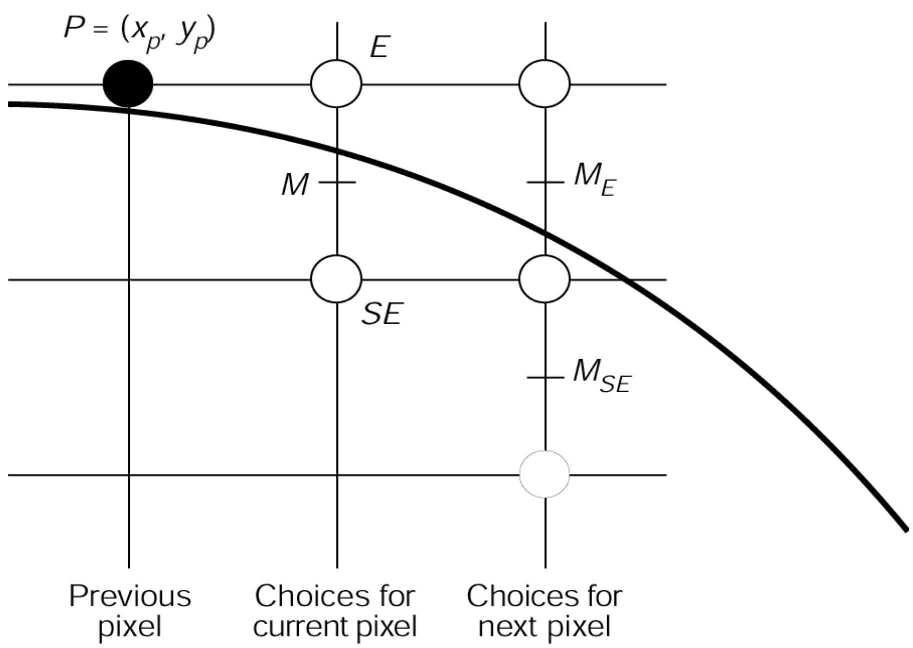
$$F(x, y) = x^2 + y^2 - R^2 = 0.$$

- Note: $F(x,y) < 0$ for points *inside* the circle, and $F(x,y) > 0$ for points *outside* the circle



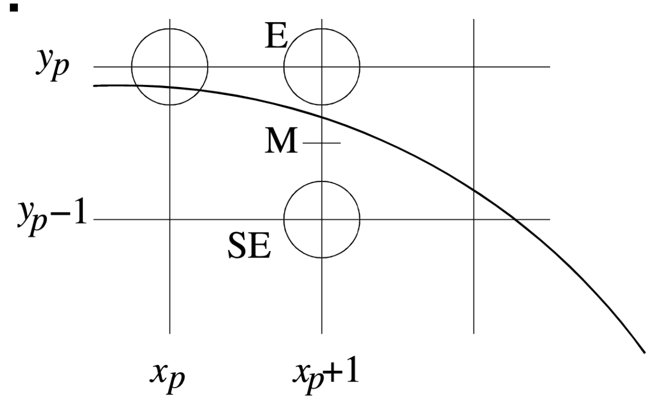
Scan Conversion of Circles

- Assume we finished pixel (x_p, y_p)
- What pixel to draw next? (going clockwise)
- Note: the slope of the circular arc is between 0 and –1
  - Hence, choice is between: *E* and *SE*
- Idea:
  If the circle passes above the midpoint *M*, then we go to *E* next, otherwise we go to *SE*



Scan Conversion of Circles

- We need a decision variable D:

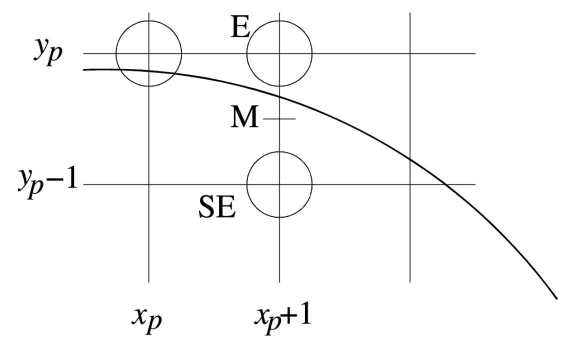


$$
\begin{aligned}
D &= F(M) = F(x_p+1, y_p-\frac{1}{2}) \\
&= (x_p+1)^2 + (y_p-\frac{1}{2})^2 - R^2.
\end{aligned}
$$

- If $D < 0$ then *M* is *below* the arc, hence the *E* pixel is closer to the line.
- If $D \geq 0$ then *M* is *above* the arc, hence the *SE* pixel is closer to the line.

Case I: When *E* is next

- What increment for computing a new *D*?
- Next midpoint is: $(x_p + 2, y_p - (1/2))$



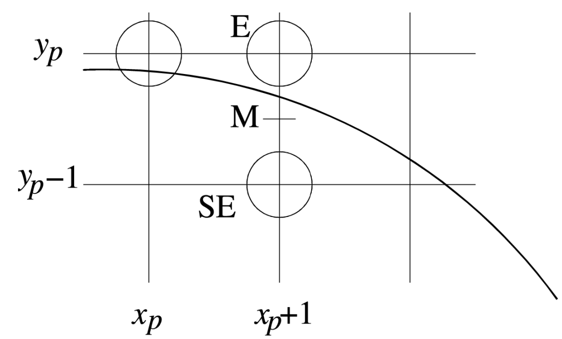
$$
\begin{aligned}
D_{new} &= F(x_p + 2, y_p - \frac{1}{2}) \\
&= (x_p + 2)^2 + (y_p - \frac{1}{2})^2 - R^2 \\
&= (x_p^2 + 4x_p + 4) + (y_p - \frac{1}{2})^2 - R^2 \\
&= (x_p^2 + 2x_p + 1) + (2x_p + 3) + (y_p - \frac{1}{2})^2 - R^2 \\
&= (x_p + 1)^2 + (2x_p + 3) + (y_p - \frac{1}{2})^2 - R^2 \\
&= D + (2x_p + 3).
\end{aligned}
$$

- Hence, increment by: $(2x_p + 3)$

Pics/Math courtesy of Dave Mount @ UMD-CP

Case II: When *SE* is next

- What increment for computing a new *D*?
- Next midpoint is: $(x_p+2, \ y_p-1-(1/2))$



$$
\begin{aligned}
D_{new} &= F(x_p+2, y_p-\frac{3}{2}) \\
&= (x_p+2)^2+(y_p-\frac{3}{2})^2-R^2 \\
&= (x_p^2+4x_p+4)+(y_p^2-3y_p+\frac{9}{4})-R^2 \\
&= (x_p^2+2x_p+1)+(2x_p+3)+(y_p^2-y_p+\frac{1}{4})+(-2y_p+\frac{8}{4})-R^2 \\
&= (x_p+1)^2+(y_p-\frac{1}{2})^2-R^2+(2x_p+3)+(-2y_p+2) \\
&= D+(2x_p-2y_p+5)
\end{aligned}
$$

- Hence, increment by: $(2x_p-2y_p+5)$

Pics/Math courtesy of Dave Mount @ UMD-CP

Scan Conversion of Circles

- How to compute the *initial* value of D:
- We start with $x = 0$ and $y = R$, so the first midpoint is at $x = 1$, $y = R-1/2$:

$$\begin{aligned} D_{init} &= F(1, R - \frac{1}{2}) \\ &= 1 + (R - \frac{1}{2})^2 - R^2 \\ &= 1 + R^2 - R + \frac{1}{4} - R^2 \\ &= \frac{5}{4} - R. \end{aligned}$$

Scan Conversion of Circles

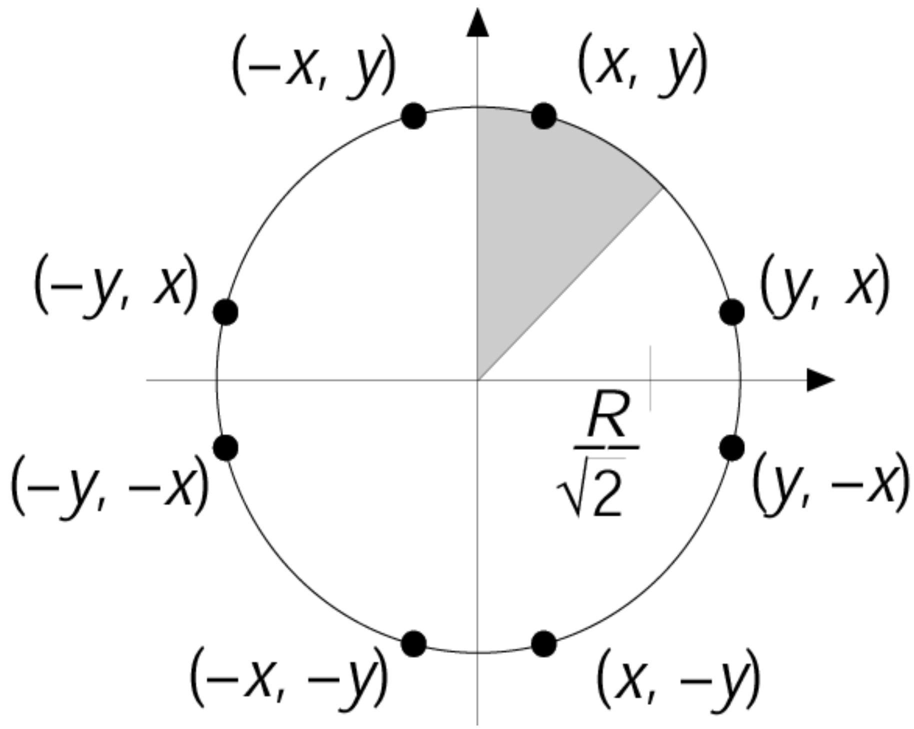
- Converting this to an integer algorithm:
  - Need only know if *D* is positive or negative
  - *D* & *R* are integers
  - Note D is incremented by an integer value
  - Therefore *D + 1/4* is positive only when *D* is positive; it is safe to drop the *1/4*
- Hence: set the initial *D* to *1 – R* (subtracting *1/4*)

Circle Scan Conversion Algorithm

- Given radius R and center (0, 0)
  - First point ➜ (0, R)
- Initial decision parameter D = 1- R
- While $x \leq y$
  - If (D < 0)
    - x++; D += 2x + 3;
  - else
    - x++; y--; D += 2(x - y) + 5
  - WritePoints(x,y)

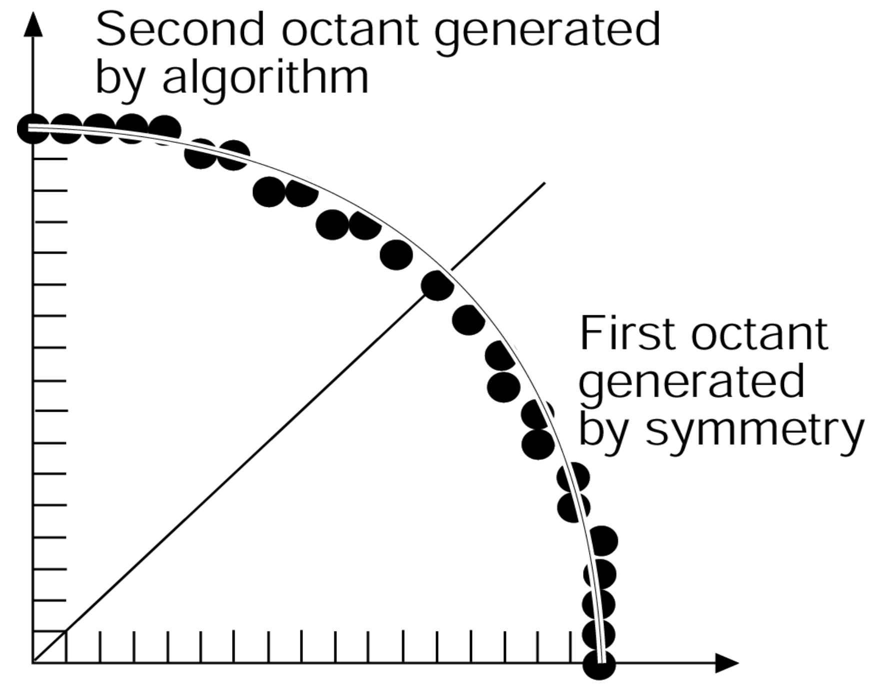
WritePoints(x,y)

- Writes pixels to the seven other octants



Clipping Circles

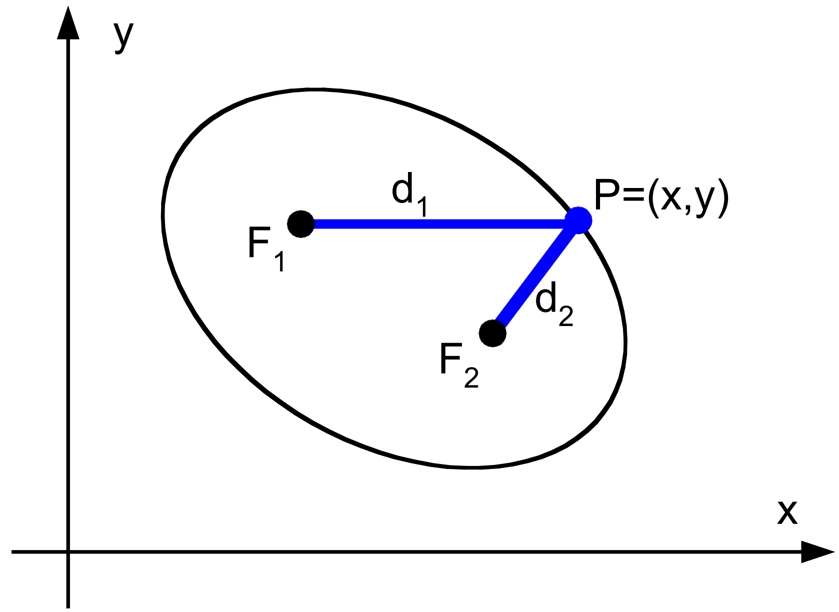
- Accept/Reject test
  - Does bounding box of the circle intersect with clipping box?
- If yes, condition pixel write on clipping box inside/outside test



Properties of Ellipses

- "Elongated circle"
- For all points on ellipse, sum of distances to foci is constant

$$d_1 + d_2 = const$$



- If $F_1=(x_1, y_1)$ and $F_2=(x_2, y_2)$ then

$$\sqrt{(x-x_1)^2 + (y-y_1)^2} + \sqrt{(x-x_2)^2 + (y-y_2)^2} = const$$

Properties of Ellipses

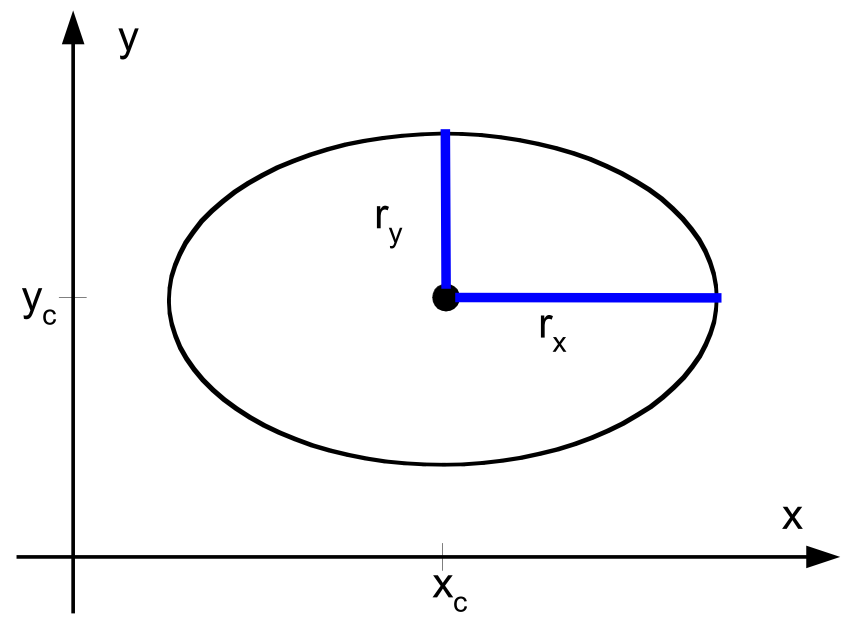
- Equation simplified if ellipse axis parallel to coordinate axis

$$\left(\frac{x - x_c}{r_x}\right)^2 + \left(\frac{y - y_c}{r_y}\right)^2 = 1$$

- Parametric form

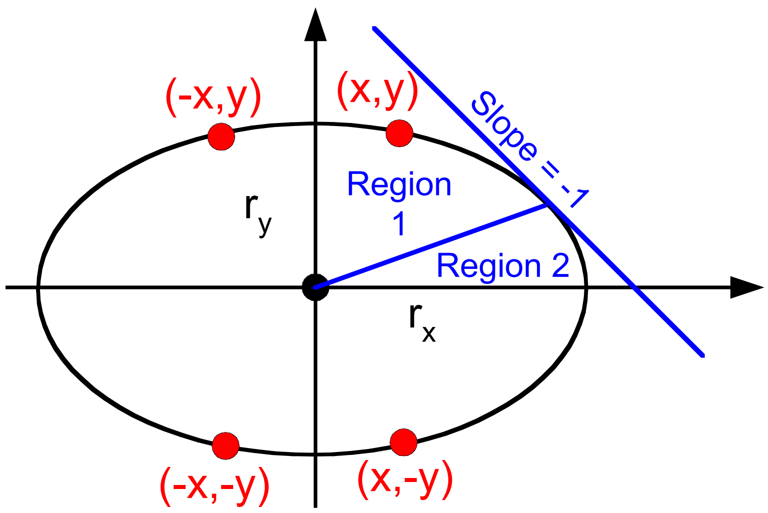
$$x = x_c + r_x \cos\theta$$

$$y = y_c + r_y \sin\theta$$



Symmetry Considerations

- 4-way symmetry
- Unit steps in x until reach region boundary
- Switch to unit steps in y



$$f(x,y) = r_y^2 x^2 + r_x^2 y^2 - r_x^2 r_y^2$$

$$\frac{dy}{dx} = -\frac{r_y^2 x}{r_x^2 y}$$

$$\frac{dy}{dx} = -1$$

$$r_y^2 x = r_x^2 y$$

- Step in x while

$$r_y^2 x < r_x^2 y$$

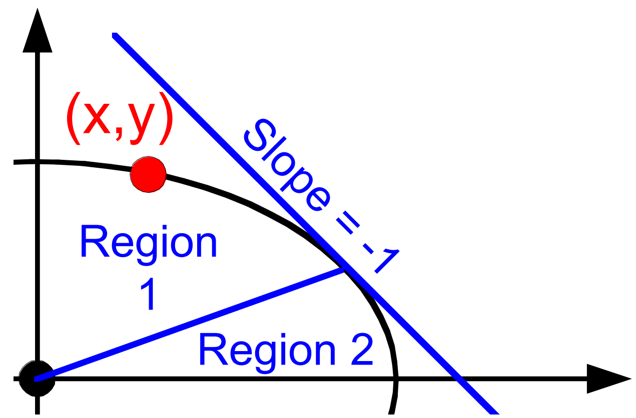
- Switch to steps in y when

$$r_y^2 x \geq r_x^2 y$$

Midpoint Algorithm (initializing)

- Similar to circles
- The initial value for region 1

$$D_{init1} = f(1, r_y - \tfrac{1}{2})$$
$$= r_y^2 - r_x^2 r_y + \tfrac{1}{4} r_x^2$$



- The initial value for region 2

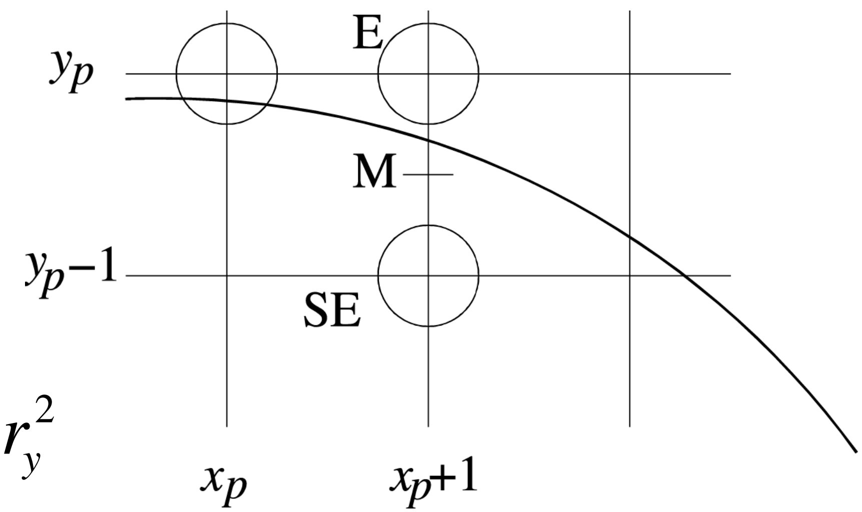
$$D_{init2} = f(x_p + \tfrac{1}{2}, y_p - 1)$$
$$= r_y^2 (x_p + \tfrac{1}{2})^2 + r_x^2 (y_p - 1)^2 - r_x^2 r_y^2$$

- We have initial values, now we need the increments

Making a Decision

- Computing the decision variable

$$D = f(x_p + 1, y_p - \tfrac{1}{2})$$
$$= r_y^2(x_p + 1)^2 + r_x^2(y_p - \tfrac{1}{2})^2 - r_x^2 r_y^2$$



- If $D < 0$ then *M* is *below* the arc, hence the *E* pixel is closer to the line.
- If $D \geq 0$ then *M* is *above* the arc, hence the *SE* pixel is closer to the line.

Computing the Increment

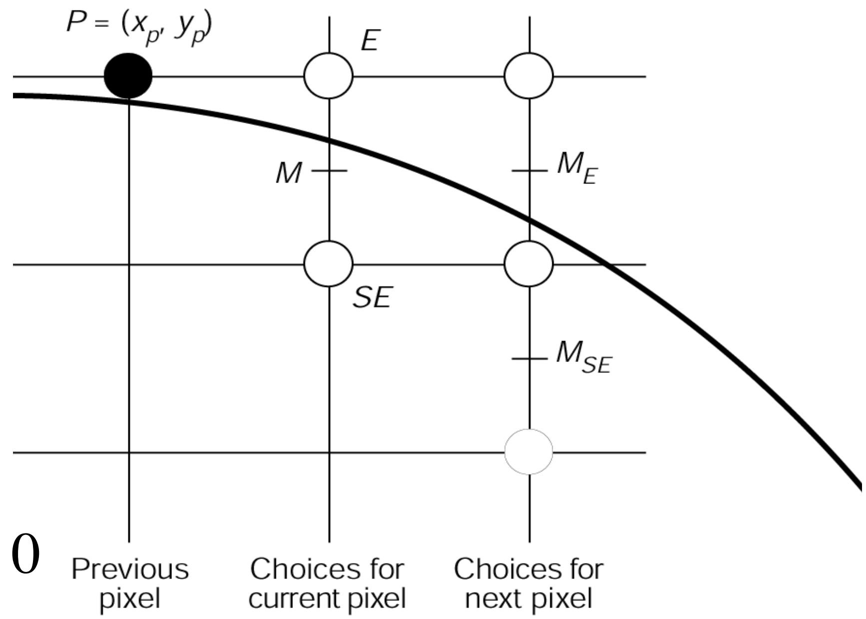
E case

$$D_{new} = f(x_p + 2, y_p - \tfrac{1}{2})$$
$$= r_y^2 (x_p + 2)^2 + r_x^2 (y_p - \tfrac{1}{2})^2 - r_x^2 r_y^2$$
$$= D_{old} + r_y^2 (2x_p + 3)$$

SE case

$$D_{new} = f(x_p + 2, y_p - \tfrac{3}{2})$$
$$= r_y^2 (x_p + 2)^2 + r_x^2 (y_p - \tfrac{3}{2})^2 - r_x^2 r_y^2$$
$$= D_{old} + r_y^2 (2x_p + 3) + r_x^2 (-2y_p + 2)$$

$$increment = \begin{cases} r_y^2 (2x_p + 3) & D_{old} < 0 \\ r_y^2 (2x_p + 3) + r_x^2 (-2y_p + 2) & D_{old} \geq 0 \end{cases}$$



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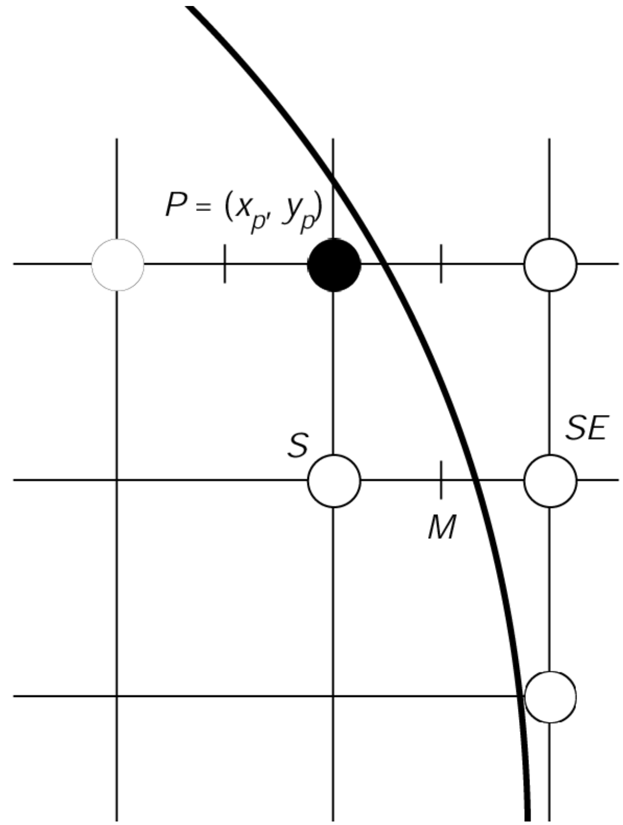
Computing the Increment in 2nd Region

- Decision variable in 2nd region

$$D = f(x_p + \tfrac{1}{2}, y_p - 1)$$
$$= r_y^2 (x_p + \tfrac{1}{2})^2 + r_x^2 (y_p - 1)^2 - r_x^2 r_y^2$$

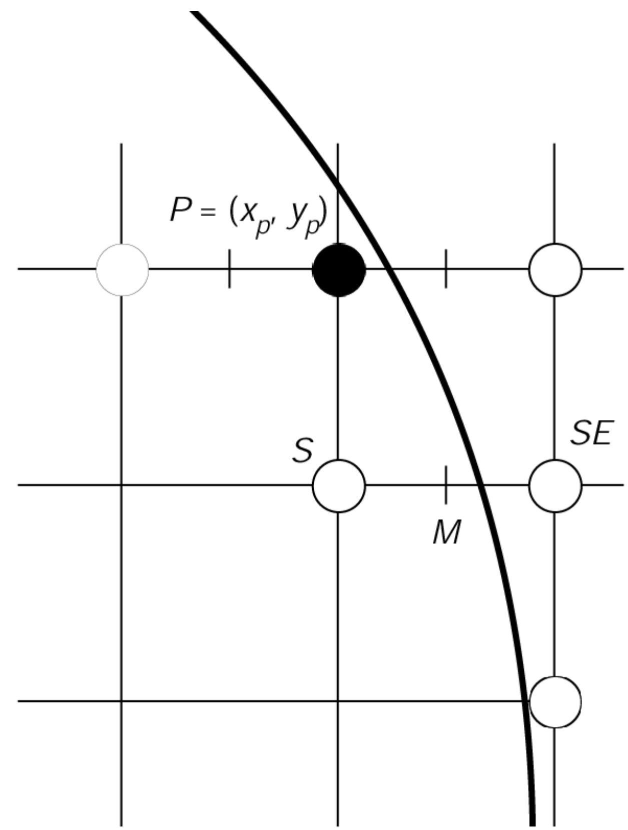


- If $D < 0$ then *M* is *left of* the arc, hence the *SE* pixel is closer to the line.
- If $D \geq 0$ then *M* is *right of* the arc, hence the *S* pixel is closer to the line.

Computing the Increment in 2nd Region

SE case

$$D_{new} = f(x_p + \tfrac{3}{2}, y_p - 2)$$
$$= r_y^2(x_p + \tfrac{3}{2})^2 + r_x^2(y_p - 2)^2 - r_x^2 r_y^2$$
$$= D_{old} + r_x^2(-2y_p + 3) + r_y^2(2x_p + 2)$$



S case

$$D_{new} = f(x_p + \tfrac{1}{2}, y_p - 2)$$
$$= r_y^2(x_p + \tfrac{1}{2})^2 + r_x^2(y_p - 2)^2 - r_x^2 r_y^2$$
$$= D_{old} + r_x^2(-2y_p + 3)$$

$$increment = \begin{cases} r_x^2(-2y_p + 3) + r_y^2(2x_p + 2) & D_{old} < 0 \\ r_x^2(-2y_p + 3) & D_{old} \geq 0 \end{cases}$$

Midpoint Algorithm for Ellipses

Region 1

Set first point to $(0,r_y)$

Set the Decision variable to

$$D_{init1} = r_y^2 - r_x^2 r_y + \tfrac{1}{4} r_x^2$$

Loop $(x = x+1)$

If $D < 0$ then pick E and

$$D \mathrel{+}= r_y^2(2x_p + 3)$$

If $D \geq 0$ then pick SE and

$$D \mathrel{+}= r_y^2(2x_p + 3) + r_x^2(-2y_p + 2)$$

Until $2r_y^2 x_k \geq 2r_x^2 y_k$

Region 2

Set first point to the last computed

Set the Decision variable to

$$D_{init2} = r_y^2(x_p + \tfrac{1}{2})^2 + r_x^2(y_p - 1)^2 - r_x^2 r_y$$

Loop $(y = y - 1)$

If $D < 0$ then pick SE and

$$D \mathrel{+}= r_y^2(2x_p + 2) + r_x^2(-2y_p + 3)$$

If $D \geq 0$ then pick S and

$$D \mathrel{+}= r_x^2(-2y_p + 3)$$

Until $y < 0$

Use symmetry to complete the ellipse