Projection Mathematics

• What is the set of transformations needed to map 3D lines/planes onto a 2D screen positioned in 3D?
• Basic procedure
  – 4D homogeneous coordinates to
  – 3D homogeneous coordinates for
    – every primitive
    – the 3D view volume

The Perspective Projection

Determining scale
• Consider
  – Point P
    – Projected onto projection plane as point P'
• Idea: compute ratios via similar triangles

The Perspective Projection

• In the x direction ratio is
  \[ \frac{x}{x'} = \frac{x}{z/d} \]
The Perspective Projection

• In the y direction ratio is

\[
\frac{z}{d} = \frac{y}{y_p} = \frac{y}{z/d}
\]

Homogeneous perspective projection matrix

\[
M_{\text{proj}} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1/d & 0
\end{bmatrix}
\]

Assumes VPN is z axis.

The Orthographic Projection

\[
M_{\text{ort}} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Assumes DOP is z axis. Just makes z component 0.

Implementing Projections
(Foley et al.)
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1. Extend 3D coordinates to homogeneous coords
2. Apply normalizing transformation, $N_{par}$ or $N_{per}$
3. Divide by W to map back down to 3D
4. Clip in 3D against canonical view volume
   - parallel or perspective view volume
5. Extend 3D coordinates back to homogeneous
6. Perform parallel projection using $M_{par}$ or
   Perform perspective projection $M_{per}$
7. Divide by W to map from homogeneous to 2D
     coordinates (division effects perspective projection)
8. Translate and scale (in 2D) to device coordinates

Canonical View Volume:
Parallel Projection

- Defined by 6 planes:
  - $x = 1$
  - $x = -1$
  - $y = 1$
  - $y = -1$
  - $z = 0$
  - $z = -1$
- Easy to clip against

Parallel Projection Pipeline
Transforming an arbitrary view volume into the canonical one

1. Translate VRP to the origin
2. Rotate so VPN becomes $z$, VUP becomes $y$ and $u$ becomes $x$
3. Shear to make direction of the projection become parallel to $z$
4. Translate and scale into a canonical view volume

1. Translate VRP to the origin
   - Simple translation $T(-VRP)$

2. Rotate
   - $R_{VPN}$ rotated to $z$
   - $R_{VUP}$ rotated to $y$
   - $R = R_{VPN} \times R_{VUP}$
   - $R_{VRP,VPN,VUP}$ in World Coordinates ($x,y,z$)

Canonical View Volume:
Perspective Projection

- Defined by 6 planes:
  - $x = z$
  - $x = -z$
  - $y = z$
  - $y = -z$
  - $z = z_{min}$
  - $z = -1$
- Easy to clip against

Top Projection
Off -Axis Projection

\[
T = \begin{bmatrix}
1 & 0 & 0 & -x_{VRP} \\
0 & 1 & 0 & -y_{VRP} \\
0 & 0 & 1 & -z_{VRP} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
2. Rotate

3. Shear

• Computing direction of projection DOP

\[
egin{bmatrix}
0 & 0 & DOP_x \\
0 & 0 & DOP_y \\
0 & 0 & DOP_z
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

PRP in VRC Coordinates (u,v,n)

3. Shear (Cont.)

• For \( Sh_{xy} \)

\[
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & sh_x \\
0 & 1 & sh_y \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

\( z' = z \), \( x' = x + zs\)h, \( y' = y + zs\)h

3. Shear (Cont)

• For \( Sh_{xy} \)

\[
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix} = \begin{bmatrix}
1 & 0 & sh_x \\
0 & 1 & sh_y \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
\]

• Find \( Sh_{xy} \) such that:

\[
DOP_x = (0,0,DOP,0) = Sh_x \cdot DOP
\]

\[
0 = DOP_x = DOP \cdot sh_x, \quad 0 = DOP_y = DOP \cdot sh_y
\]

\[
sh_x = \frac{DOP_x}{DOP}, \quad sh_y = \frac{DOP_y}{DOP}
\]

3. Shear (Finally)

• Given

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\frac{1}{2}(sh_x + k) - \frac{1}{2}(sh_x) \\
\frac{1}{2}(sh_y + k) - \frac{1}{2}(sh_y) \\
\frac{1}{2}(sh_z + k) - \frac{1}{2}(sh_z)
\end{bmatrix}
\]

• We can compute the \( Sh_{xy} \) matrix

\[
\begin{bmatrix}
1 & 0 & \frac{1}{2}(sh_x + k) - \frac{1}{2}(sh_x) & \frac{1}{2}(sh_y + k) - \frac{1}{2}(sh_y) & \frac{1}{2}(sh_z + k) - \frac{1}{2}(sh_z) \\
0 & 1 & \frac{1}{2}(sh_x + k) - \frac{1}{2}(sh_x) & \frac{1}{2}(sh_y + k) - \frac{1}{2}(sh_y) & \frac{1}{2}(sh_z + k) - \frac{1}{2}(sh_z) \\
0 & 0 & 1 & \frac{1}{2}(sh_x + k) - \frac{1}{2}(sh_x) & \frac{1}{2}(sh_y + k) - \frac{1}{2}(sh_y) & \frac{1}{2}(sh_z + k) - \frac{1}{2}(sh_z)
\end{bmatrix}
\]
4. Translate and Scale

- Translate the center of the volume to the origin
- Scaling to 2x2
- Needed for 3D clipping

Foley et al. is wrong!

\[ T_{\text{par}} = \begin{bmatrix} 1 & 0 & 0 & -\text{vrp}_x \\ 0 & 1 & 0 & -\text{vrp}_y \\ 0 & 0 & 1 & -\text{vrp}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

Parallel Projection View
Volume Transformation

\[ N_{\text{par}} = (S_{\text{par}} \cdot (T_{\text{par}} \cdot (SH_{\text{par}} \cdot (R \cdot T(-\text{VRP})))) ) \]

- Apply to all model vertices
- \( P' = N_{\text{par}} P \)

Parallel Projection Summary

Perspective Projection Pipeline

Transforming an arbitrary view volume into the canonical one

1. Translate VRP to the origin
2. Rotate so VPN becomes \( z \)
   VUP becomes \( y \) and \( w \) becomes \( x \)
3. Translate COP to origin
4. Shear so volume centerline becomes \( z \) axis
5. Scale into a canonical view volume for clipping

1. Translate VRP to the origin

- Simple translation \( T(-\text{VRP}) \)

\[ T_{\text{VRP}} = \begin{bmatrix} 1 & 0 & 0 & -\text{vrp}_x \\ 0 & 1 & 0 & -\text{vrp}_y \\ 0 & 0 & 1 & -\text{vrp}_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \]
3. Translate

- Simple translation \( T(\text{PRP}) \)

\[
T = \begin{bmatrix}
1 & 0 & 0 & -\text{PRP}_x \\
0 & 1 & 0 & -\text{PRP}_y \\
0 & 0 & 1 & -\text{PRP}_z \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

PRP in VRC
Coordinates (x,y,z)

5. Scaling

- We can define VRP after transformation as
  \( \text{VRP}' = \text{VRP} \times T(\text{PRP}) \times \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \)
- \( \text{vRP}' = \text{PRP} \) since shear does not affect z coordinates
  - First, we scale differentially in x and y to set plane slopes to 1 and -1
  - Second, we scale uniformly by \( \frac{1}{\text{max}} \)

5. Scaling (Cont)

- We can combine the transformations

4. Shear

- Goal is to transform the center line to the z axis
- Same as parallel projection
- Shear matrix is the same

5. Scaling (Cont)

- Back clipping plane is \( z = -I \)
- Front clipping plane is \( z = \text{PRP}_z \)
- Projection plane is \( z = -\text{PRP}_z \)
Perspective Projection Pipeline

\[ N_{\text{per}} = (S_{\text{per}} \cdot (S_{H_{\text{par}}} \cdot (T\cdot\text{PRP}) \cdot (R \cdot T\cdot\text{VRP}))) \]

- Apply to all model vertices
- \[ P' = N_{\text{per}} \cdot P \]

Summary of 3D Transforms

- We know how to take any projection and convert it into a canonical View Volume
- 3D edges can be clipped against it and projected onto screen

Implementing Projections Without 3D Clipping

1. Extend 3D coordinates to homogeneous coordinates
2. Apply normalizing transformation, \( N_{\text{par}} \) or \( N_{\text{per}} \)
3. Perform trivial reject test with view volume
4. Perform parallel projection or perform perspective projection
5. Clip against 2D “world” (view plane) window
6. Translate and scale (in 2D) to device coordinates (i.e. into viewport)
7. Draw/Fill polygons

Transformed Window

- Parallel Projection
  - \((-1,-1) \rightarrow (1,1)\)
- Perspective Projection
  - \((-\|z_{\text{proj}}\|,-\|z_{\text{proj}}\|) \rightarrow (\|z_{\text{proj}}\|, \|z_{\text{proj}}\|)\)
  - \((-d,-d) \rightarrow ((d, d))\)

- Use these for world window parameters in viewport mapping and 2D clipping

Programming assignment 4

- Read SMF file
- Implement parallel projection
- Implement perspective projection
- Output projected and clipped polygon edges