


Geometry

CS 432 Interactive Computer Graphics
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1




Objectives

- Introduce the elements of geometry
 - Scalars
 - Vectors
 - Points
- Develop mathematical operations among them in a coordinate-free manner
- Define basic primitives
 - Line segments
 - Polygons

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2




Basic Elements

- Geometry is the study of the relationships among objects in an n -dimensional space
 - In computer graphics, we are interested in objects that exist in three dimensions
- Want a minimum set of primitives from which we can build more sophisticated objects
- We will need three basic elements
 - Scalars
 - Vectors
 - Points

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3




Coordinate-Free Geometry

- When we learned simple geometry, most of us started with a Cartesian approach
 - Points were at locations in space $p=(x,y,z)$
 - We derived results by algebraic manipulations involving these coordinates
- This approach was nonphysical
 - Physically, points exist regardless of the location of an arbitrary coordinate system
 - Most geometric results are independent of the coordinate system
 - Example Euclidean geometry: two triangles are identical if two corresponding sides and the angle between them are identical

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4




Scalars

- Need three basic elements in geometry
 - Scalars, Vectors, Points
- Scalars can be defined as members of sets which can be combined by two operations (addition and multiplication) obeying some fundamental axioms (associativity, commutivity, inverses)
- Examples include the real and complex number systems under the ordinary rules with which we are familiar
- Scalars alone have no geometric properties


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Vectors

- Physical definition: a vector is a quantity with two attributes
 - Direction
 - Magnitude
- Examples include
 - Force
 - Velocity
 - Directed line segments
 - Most important example for graphics
 - Can map to other types

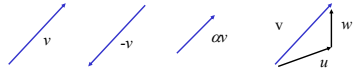


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Vector Operations

- Every vector has an inverse
 - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
 - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
 - Use head-to-tail axiom



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7

Linear Vector Spaces

- Mathematical system for manipulating vectors
- Operations
 - Scalar-vector multiplication $u = \alpha v$
 - Vector-vector addition: $w = u + v$
- Expressions such as
 - $v = u + 2w - 3r$

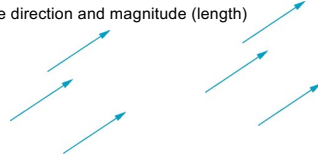
Make sense in a vector space

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8

Vectors Lack Position

- These vectors are identical
 - Same direction and magnitude (length)
- Vectors spaces insufficient for geometry
 - Need points

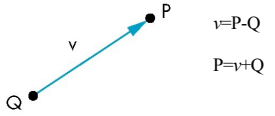


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9

Points

- Location in space
- Operations allowed between points and vectors
 - Point-point subtraction yields a vector
 - Equivalent to point-vector addition



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10

Affine Spaces

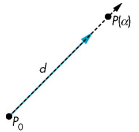
- Point + a vector space
- Operations
 - Vector-vector addition
 - Scalar-vector multiplication
 - Point-vector addition
 - Scalar-scalar operations
- For any point define
 - $1 \cdot P = P$
 - $0 \cdot P = \mathbf{0}$ (zero vector)

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11

Lines

- Consider all points of the form
 - $P(\alpha) = P_0 + \alpha d$
 - Set of all points that pass through P_0 in the direction of the vector d



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Parametric Form

- This form is known as the parametric form of the line
 - More robust and general than other forms
 - Extends to curves and surfaces
- Two-dimensional forms
 - Explicit: $y = mx + h$
 - Implicit: $ax + by + c = 0$
 - Parametric:

$$x(\alpha) = \alpha x_0 + (1-\alpha)x_1$$

$$y(\alpha) = \alpha y_0 + (1-\alpha)y_1$$

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Rays and Line Segments

- If $\alpha > 0$, then $P(\alpha)$ is the ray leaving P_0 in the direction \mathbf{d}
- If we use two points to define \mathbf{v} , then

$$P(\alpha) = Q + \alpha(R-Q) = Q + \alpha\mathbf{v}$$

$$= \alpha R + (1-\alpha)Q$$
- For $0 \leq \alpha \leq 1$ we get all the points on the line segment joining R and Q

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14

Convexity

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object

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Affine Sums

- Consider the "sum"

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_n P_n$$
- If

$$\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$$
 in which case we have the *affine sum* of the points P_1, P_2, \dots, P_n
- If, in addition, $\alpha_i \geq 0$, we have the *convex hull* of P_1, P_2, \dots, P_n

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Convex Hull

- Smallest convex object containing P_1, P_2, \dots, P_n
- Formed by "shrink wrapping" points

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Curves and Surfaces

- Curves are one parameter entities of the form $P(\alpha)$ where the function is nonlinear
- Surfaces are formed from two-parameter functions $P(\alpha, \beta)$
 - Linear functions give planes and polygons

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Planes

- A plane can be defined by a point and two vectors or by three points

$P(\alpha, \beta) = R + \alpha u + \beta v$
 $P(\alpha, \beta) = R + \alpha(Q - R) + \beta(P - R)$

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19

Triangles

convex sum of P and Q
 $T(\alpha, \beta)$
convex sum of $S(\alpha)$ and R
 $S(\alpha)$
P Q R
for $0 \leq \alpha, \beta \leq 1$, we get all points in triangle

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20

Barycentric Coordinates

Triangle is convex so any point inside can be represented as an affine sum

$$P(\alpha_1, \alpha_2, \alpha_3) = \alpha_1 P + \alpha_2 Q + \alpha_3 R$$

where

$$\alpha_1 + \alpha_2 + \alpha_3 = 1$$

$$\alpha_i \geq 0$$

The representation is called the **barycentric coordinate** representation of P

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21

Normals

- Every plane has a vector n normal (perpendicular, orthogonal) to it
- From point-two vector form $P(\alpha, \beta) = R + \alpha u + \beta v$, we know we can use the cross product to find $n = u \times v$ and the equivalent form $(P(\alpha) - P) \cdot n = 0$

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22

Representation

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Objectives

- Introduce concepts such as dimension and basis
- Introduce coordinate systems for representing vectors spaces and frames for representing affine spaces
- Discuss change of frames and bases
- Introduce homogeneous coordinates

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24


 **Linear Independence**

- A set of vectors v_1, v_2, \dots, v_n is *linearly independent* if

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \text{ iff } \alpha_1 = \alpha_2 = \dots = 0$$
- If a set of vectors is linearly independent, we cannot represent one in terms of the others
- If a set of vectors is linearly dependent, at least one can be written in terms of the others

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25


 **Dimension**

- In a vector space, the maximum number of linearly independent vectors is fixed and is called the *dimension* of the space
- In an n -dimensional space, any set of n linearly independent vectors form a *basis* for the space
- Given a basis v_1, v_2, \dots, v_n , any vector v can be written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$
 where the $\{\alpha_i\}$ are unique

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26

 **Representation**

- Until now we have been able to work with geometric entities without using any frame of reference, such as a coordinate system
- Need a frame of reference to relate points and objects to our physical world.
 - For example, where is a point? Can't answer without a reference system
 - World coordinates
 - Camera coordinates

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27


 **Coordinate Systems**

- Consider a basis v_1, v_2, \dots, v_n
- A vector is written $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$
- The list of scalars $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is the *representation* of v with respect to the given basis
- We can write the representation as a row or column array of scalars

$$\mathbf{a} = [\alpha_1 \ \alpha_2 \ \dots \ \alpha_n]^T = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

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28

 **Example**

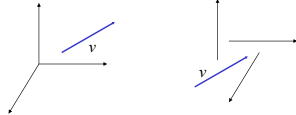
- $v = 2v_1 + 3v_2 - 4v_3$
- $\mathbf{a} = [2, 3, -4]^T$
- Note that this representation is with respect to a particular basis
- For example, in WebGL we start by representing vectors using the object basis but later the system needs a representation in terms of the camera or eye basis

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29

 **Coordinate Systems**

• Which is correct?



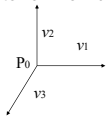
- Both are because vectors have no fixed location

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30

Frames

- A coordinate system is insufficient to represent points
- If we work in an affine space we can add a single point, the *origin*, to the basis vectors to form a *frame*



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Representation in a Frame

- Frame determined by (P_0, v_1, v_2, v_3)
- Within this frame, every vector can be written as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$
- Every point can be written as

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$$

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Confusing Points and Vectors

Consider the point and the vector

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$$

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

They appear to have the similar representations

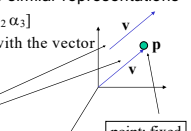
$$p = [\beta_1 \beta_2 \beta_3] \quad v = [\alpha_1 \alpha_2 \alpha_3]$$

which confuses the point with the vector

A vector has no position

Vector can be placed anywhere

point: fixed



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Homogeneous Coordinates

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A Single Representation

If we define $0 \cdot P = 0$ and $1 \cdot P = P$ then we can write

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = [\alpha_1 \alpha_2 \alpha_3 0] [v_1 v_2 v_3 P_0]^T$$

$$P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = [\beta_1 \beta_2 \beta_3 1] [v_1 v_2 v_3 P_0]^T$$

Thus we obtain the four-dimensional *homogeneous coordinate* representation

$$v = [\alpha_1 \alpha_2 \alpha_3 0]^T$$

$$p = [\beta_1 \beta_2 \beta_3 1]^T$$

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Homogeneous Coordinates

The homogeneous coordinates form for a three dimensional point $[x y z]$ is given as

$$p = [x' y' z' w]^T = [wx wy wz w]^T$$

We return to a three dimensional point (for $w \neq 0$) by

$$x \leftarrow x' / w$$

$$y \leftarrow y' / w$$

$$z \leftarrow z' / w$$

If $w=0$, the representation is that of a vector

Note that homogeneous coordinates replaces points in three dimensions by lines through the origin in four dimensions

For $w=1$, the representation of a point is $[x y z 1]$

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Homogeneous Coordinates and Computer Graphics

- Homogeneous coordinates are key to all computer graphics systems
 - All standard transformations (rotation, translation, scaling) can be implemented with matrix multiplications using 4 x 4 matrices
 - Hardware pipeline works with 4 dimensional representations
 - For orthographic viewing, we can maintain $w=0$ for vectors and $w=1$ for points
 - For perspective we need a *perspective division*

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37

Change of Coordinate Systems

- Consider two representations of the same vector with respect to two different bases. The representations are

$$\mathbf{a} = [\alpha_1 \ \alpha_2 \ \alpha_3]$$

$$\mathbf{b} = [\beta_1 \ \beta_2 \ \beta_3]$$

where

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = [\alpha_1 \ \alpha_2 \ \alpha_3] [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]^T$$

$$= \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \beta_3 \mathbf{u}_3 = [\beta_1 \ \beta_2 \ \beta_3] [\mathbf{u}_1 \ \mathbf{u}_2 \ \mathbf{u}_3]^T$$

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38

Representing second basis in terms of first

Each of the basis vectors, $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, are vectors that can be represented in terms of the first basis

$$\mathbf{u}_1 = \gamma_{11}\mathbf{v}_1 + \gamma_{12}\mathbf{v}_2 + \gamma_{13}\mathbf{v}_3$$

$$\mathbf{u}_2 = \gamma_{21}\mathbf{v}_1 + \gamma_{22}\mathbf{v}_2 + \gamma_{23}\mathbf{v}_3$$

$$\mathbf{u}_3 = \gamma_{31}\mathbf{v}_1 + \gamma_{32}\mathbf{v}_2 + \gamma_{33}\mathbf{v}_3$$

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39

Matrix Form

The coefficients define a 3 x 3 matrix

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

and the bases can be related by

$$\mathbf{a} = \mathbf{M}^T \mathbf{b}$$

see text for numerical examples

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40

Change of Frames

- We can apply a similar process in homogeneous coordinates to the representations of both points and vectors

Consider two frames:
 $(P_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$
 $(Q_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$

- Any point or vector can be represented in either frame
- We can represent $Q_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ in terms of $P_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

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41

Representing One Frame in Terms of the Other

Extending what we did with change of bases

$$\mathbf{u}_1 = \gamma_{11}\mathbf{v}_1 + \gamma_{12}\mathbf{v}_2 + \gamma_{13}\mathbf{v}_3$$

$$\mathbf{u}_2 = \gamma_{21}\mathbf{v}_1 + \gamma_{22}\mathbf{v}_2 + \gamma_{23}\mathbf{v}_3$$

$$\mathbf{u}_3 = \gamma_{31}\mathbf{v}_1 + \gamma_{32}\mathbf{v}_2 + \gamma_{33}\mathbf{v}_3$$

$$Q_0 = \gamma_{41}\mathbf{v}_1 + \gamma_{42}\mathbf{v}_2 + \gamma_{43}\mathbf{v}_3 + \gamma_{44}P_0$$

defining a 4 x 4 matrix

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$

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42

Working with Representations

Within the two frames any point or vector has a representation of the same form

$\mathbf{a} = [\alpha_1 \ \alpha_2 \ \alpha_3 \ \alpha_4]$ in the first frame
 $\mathbf{b} = [\beta_1 \ \beta_2 \ \beta_3 \ \beta_4]$ in the second frame

where $\alpha_4 = \beta_4 = 1$ for points and $\alpha_4 = \beta_4 = 0$ for vectors and

$\mathbf{a} = \mathbf{M}^T \mathbf{b}$

The matrix \mathbf{M} is 4 x 4 and specifies an affine transformation in homogeneous coordinates

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43

Transformations to Change Coordinate Systems

- 4 coordinate systems
- 1 point P

$M_{1 \leftarrow 2} = T(4,2)$
 $M_{2 \leftarrow 3} = T(2,3) \cdot S(0.5,0.5)$
 $M_{3 \leftarrow 4} = T(6.7,1.8) \cdot R(45^\circ)$

$M_{i \leftarrow k} = M_{i \leftarrow j} \cdot M_{j \leftarrow k}$

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44

The World and Camera Frames

- When we work with representations, we work with n-tuples or arrays of scalars
- Changes in frame are then defined by 4 x 4 matrices
- In WebGL, the base frame that we start with is the world frame
- Eventually we represent entities in the camera frame by changing the world representation using the model-view matrix
- Initially these frames are the same ($\mathbf{M} = \mathbf{I}$)

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46

Moving the Camera

If objects are on both sides of $z=0$, we must move camera frame

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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47