Review of Complex Numbers

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Objective: To review the basic properties of complex numbers.

- Definition
- Arithmetic and norm
- Polar coordinates and Euler’s identity
- Roots of unity
Definition

• Let \( i = \sqrt{-1} \), i.e. a root of \( x^2 + 1 \)

• The set of complex numbers = \{a+bi, a, b \text{ real numbers}\}

• The complex number \( a+bi \) can be thought of as the pair \((a,b)\) and can be drawn in the complex plane, where the x axis is called the real axis and the y axis is called the imaginary axis.

• Any point is a linear combination of \( i = (0,1) \) and \( 1 = (1,0) \), i.e.

\[
ia + bi = a(1,0) + b(0,1)\]
Arithmetic and Norm

• \((a+bi) + (c+di) = (a+c) + (b+d)i\)
• \((a+bi) * (c+di) = ac + adi + bci + b^2d = ac – bd + (ad+bc)i\)

• The complex conjugate of \(a+bi = a-bi\)
• The norm of \(a+bi\) is the real number \(= (a+bi)(a-bi) = a^2+b^2\)

• The inverse of the complex number \(a+bi = a-bi/(a^2+b^2). \) By the definition of the norm it is clear that \((a+bi)(a-bi/(a^2+b^2)) = 1.\)
Polar Coordinates

- Any point \( z = (a,b) \) in the plane can be represented by a distance \( r \) and an angle \( \theta \). The pair \((r,\theta)\) is called the polar coordinates of the point \( z \).

\[
r = \sqrt{a^2 + b^2}
\]

\[
\theta = \arctan(b/a)
\]
Euler’s Identity

• \( e^{i\theta} = \cos(\theta) + \sin(\theta)i \)

• Since \( \cos^2(\theta) + \sin^2(\theta) = 1 \), \( \text{Norm}(e^{i\theta}) = 1 \), and the complex number \( e^{i\theta} \) lies on the unit circle, i.e. the collection of complex numbers \( a + bi \) such that the point \( (a, b) \) satisfies the equation \( a^2 + b^2 = 1 \) (circle with radius 1).

• By Euler’s identity and polar coordinates an arbitrary complex number \( a + bi = re^{i\theta} \), where

\[
r = \sqrt{a^2 + b^2}
\]

\[
\theta = \arctan(b / a)
\]
Roots of Unity

- An nth root of unity is a complex number $\omega$ such that $\omega^n = 1$. An nth root of unity $\omega$ is a primitive nth root of unity if $\omega^j \neq 1$ for $0 < j < n$.

- $\omega = e^{2\pi i/n} = \cos(2\pi/n) + \sin(2\pi/n)i$, is an nth root of unity since $\omega^n = (e^{2\pi i/n})^n = e^{2\pi i} = \cos(2\pi) + \sin(2\pi)i = 1$

- Moreover, $\omega = e^{2\pi i/n}$ is a primitive nth root of unity since $\omega^j = e^{2\pi ij/n} = \cos(2\pi j/n) + \sin(2\pi j/n)i \neq 1$.

- Since an nth root of unity satisfies the equation $x^n - 1 = 0$, there are at most $n$ nth roots of unity. Since $\omega^j$, is also an nth root of unity, powers of a primitive nth root of unity generate all nth roots of unity.
Roots of Unity (cont)

• Multiplication by $\omega = e^{2\pi i / n}$ rotates the number by $2\pi / n$ radians, and successive powers rotate around the circle. Powers of a primitive $n$th root of unity divide the unit circle into $n$ equal arcs of $2\pi / n$ radians.

• E.G. $n = 3$ (cube roots of unity)