Recursion and Induction

• Themes
  – Recursion
  – Recurrence Definitions
  – Recursive Relations
  – Induction (prove properties of recursive programs and objects defined recursively)

• Examples
  – Tower of Hanoi
  – Gray Codes
  – Hypercube
Recursion & Recurrence Relations

- Very handy for defining functions and data types simply:
  - Consider the \( n \)th Fibonacci number, \( F_n \):
    - 0 if \( n = 0 \), 1 if \( n=1 \)
    - \( F_{n-1} + F_{n-2} \), for all \( n>1 \)

- Very handy when a large problem can be broken in similar (but smaller) problems
  - We’ll look at the Towers of Hanoi in a moment
Who needs Induction?

Useful for reasoning precisely about algorithms, and programs

– Correctness
  • Prove that a program correctly computes its result

– Performance
  • How does the runtime of a particular algorithm grow vs. the inputs (number and/or size)?
Induction & Recursion

• Very similar notions. They have exactly the same roots

• Inductive proofs apply in a very natural way to recursive algorithms, and recurrence relations
Tower of Hanoi

- There are three towers
- 64 gold disks, with decreasing sizes, placed on the first tower
- You need to move the stack of disks from one tower to another, one disk at a time
- Larger disks can not be placed on top of smaller disks
- The third tower can be used to temporarily hold disks
Tower of Hanoi

• Assume one disk can be moved in 1 second
  How long would it take to move 64 disks? N disks?

• To create an algorithm to solve this problem, it is convenient to generalize the problem to the “N-disk” problem, where in our case N = 64.
Recursive Solution
Recursive Solution
Recursive Solution
Recursive Solution
Tower of Hanoi
Tower of Hanoi
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Recursive Algorithm

(defun move (from to)
  (list (concatenate 'string from to)))

(defun hanoi (n from to using)
  (if (equal n 1)
      (move from to)
      (append (hanoi (- n 1) from using to)
              (move from to)
              (hanoi (- n 1) using to from))))
Induction

• To prove a statement $S(n)$ for positive integers $n$
  – Need a base case (typically $n=0$ or $n=1$). Prove that $S(0)$ or $S(1)$ is true
  – Assume $S(n)$ is true [inductive hypothesis]
  – Prove that $S(n+1)$ is true. You’ll need to use the hypothesis in this step, or you did something wrong
Correctness

• Use induction to prove that the recursive algorithm solves the Tower of Hanoi problem.

• Let $H(n,a,b,c) =$ property that $(\text{hanoi } n \ a \ b \ c)$ moves $n$ disks from tower $a$ to $b$ using tower $c$ without placing larger disks on top of smaller disks
Correctness

• **Base case:**
  • $H(1,a,b,c)$ works since \((\text{hanoi } 1 \ a \ b \ c) = (\text{move } a \ b)\)

• **Inductive Hypothesis (IH):**
  • Assume $H(n,a,b,c)$ is correct for arbitrary $a \ b \ c$

• **Show IH $\Rightarrow H(n+1,a,b,c)$**
  • The first recursive call correctly moves $n$ disks from tower $a$ to tower $c$ [by IH]
  • Move moves the largest disk to the empty tower $b$ (all other disks on tower $c$)
  • The second recursive call correctly moves $n$ disks from tower $c$ to tower $b$ on top of largest disk
Cost

• Show that the number of moves $M(n)$ required by the algorithm to solve the n-disk problem satisfies the recurrence relation
  - $M(n) = 2M(n-1) + 1$
  - $M(1) = 1$

• This can be done inductively, and it would be very similar to the last proof.
Cost

• We’d like to find a closed form, a formula that is not a recurrence relation

• We can do this a couple ways:
  – We can guess at the answer (and then prove it, of course)
  – We can unwind the recurrence (still need to prove it)
Guess and Prove

- Calculate $M(n)$ for small $n$ and look for a pattern.
- Guess the result and prove your guess correct using induction.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$M(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>15</td>
</tr>
<tr>
<td>5</td>
<td>31</td>
</tr>
</tbody>
</table>
Proof by Induction

• **Base case:** \( M(1) = 1 = 2^1 - 1 \)

• **Inductive Hypothesis (IH)**
  • Assume \( M(n) = 2^n - 1 \)

• **Show IH \( \Rightarrow M(n+1) = 2^{n+1} - 1 \)**
  • \( M(n+1) = 2M(n) + 1 \)
  • \( = 2(2^n - 1) + 1 \) [by IH]
  • \( = 2 \times 2^n - 1 = 2^{n+1} - 1 \)
Substitution Method

• Unwind recurrence, by repeatedly replacing \( M(n) \) by the r.h.s. of the recurrence until the base case is encountered.

\[
M(n) = 2M(n-1) + 1
= 2\times[2\times M(n-2) + 1] + 1 = 2^2 \times M(n-2) + 1 + 2
= 2^2 \times [2\times M(n-3) + 1] + 1 + 2
= 2^3 \times M(n-3) + 1 + 2 + 2^2
\]
Geometric Series

• After k steps: (prove by induction on k)
  \[ M(n) = 2^k \cdot M(n-k) + 1+2 + 2^2 + \ldots + 2^{k-1} \]

• Base case, M(1), encountered when n-k=1
  \[ \Rightarrow k = n-1 \]

• Substituting in, to get rid of all of the k’s:
  \[ M(n) = 2^{n-1} \cdot M(1) + 1+2 + 2^2 + \ldots + 2^{n-2} \]
  \[ = 1 + 2 + \ldots + 2^{n-1} = \sum_{i=0}^{n-1} 2^i = 2^n - 1 \]

• Use induction to reprove result for M(n) using this sum. Generalize by replacing 2 by x.
Induction

• After k steps: (prove by induction on k)
  \[ M(n) = 2^k \times M(n-k) + 1 + 2 + 2^2 + \ldots + 2^{k-1} \quad (1 \leq k < n) \]

• Base case:
  • Using recurrence after 1 step \( M(n) = 2M(n-1) + 1 \)

• Induction Hypothesis
  \[ M(n) = 2^k \times M(n-k) + 1 + 2 + 2^2 + \ldots + 2^{k-1} \]

• Show IH \( \Rightarrow \)
  \[ M(n) = 2^{k+1} \times M(n-(k+1)) + 1 + 2 + 2^2 + \ldots + 2^k \]
Induction

• Show IH \( \Rightarrow \)

\[
M(n) = 2^{k+1} \times M(n-(k+1)) + 1 + 2 + 2^2 + \ldots + 2^k
\]

\[
M(n) = 2^k \times M(n-k) + 1 + 2 + 2^2 + \ldots + 2^{k-1} \quad [\text{IH}]
\]

\[
= 2^k \times (2M(n-k)+1) + 1 + 2 + 2^2 + \ldots + 2^{k-1} \quad [\text{recurrence}]
\]

\[
= 2^k \times 2M(n-k-1) + 2^k + 1 + 2 + 2^2 + \ldots + 2^{k-1}
\]

\[
= 2^{k+1} \times M(n-(k+1)) + 1 + 2 + 2^2 + \ldots + 2^k
\]
Geometric Series

• Show \( 1 + 2 + \ldots + 2^{n-1} = \sum_{i=0}^{n-1} 2^i = 2^n - 1 \)

• Base case: \( \sum_{i=0}^{0} 2^i = 1 = 2 - 1 \)

• Inductive Hypothesis \( \sum_{i=0}^{n-1} 2^i = 2^n - 1 \)

• Show \( \sum_{i=0}^{n} 2^i = 2^n + \sum_{i=0}^{n-1} 2^i = 2^n + (2^n - 1) = 2 \times 2^n - 1 = 2^{n+1} - 1 \)

• Generalize by replacing 2 by x.
Worst Case

• Is it possible to solve Hanoi in fewer than $2^n-1$ moves?

• Let $M(n)$ be the fewest moves required to solve $H(n,a,b,c)$

• Claim $M(n) \geq 2M(n-1) + 1$ [why?]
  • Use induction to show $M(n) \geq 2^n-1$
Induction Examples

Prove:

• $1 + 3 + 5 + \ldots + (2n-1) = n^2$
• $4n < 2^n$, $\forall \ n \geq 5$
• $7 \mid 8^n - 1$
• $2^n < n!$, $\forall \ n \geq 4$
Gray Code

• An n-bit Gray code is a 1-1 onto mapping from $[0..2^n-1]$ such that the binary representation of consecutive numbers differ by exactly one bit.

• Invented by Frank Gray for a shaft encoder - a wheel with concentric strips and a conducting brush which can read the number of strips at a given angle. The idea is to encode $2^n$ different angles, each with a different number of strips, corresponding to the n-bit binary numbers.
Shaft Encoder (Counting Order)

Consecutive angles can have an abrupt change in the number of strips (bits) leading to potential detection errors.
Shaft Encoder (Gray Code)

Since a Gray code is used, consecutive angles have only one change in the number of strips (bits).
Binary-Reflected Gray Code

• $G_1 = [0,1]$
• $G_n = [0G_{n-1},1\overline{G_{n-1}}]$, $\overline{G}$ \Rightarrow reverse order $\equiv$ complement leading bit

• $G_2 = [0G_1,1\overline{G_1}] = [00,01,11,10]$
• $G_3 = [0G_2,1\overline{G_2}] = [000,001,011,010,110,111,101,100]$
• Use induction to prove that this is a Gray code
Iterative Formula

• Let $G_n(i)$ be a function from $[0,\ldots,2^n-1]$

• $G_n(i) = i \oplus (i >> 1)$ [exclusive or of $i$ and $i/2$]
  – $G_2(0) = 0, G_2(1) = 1, G_2(2) = 3, G_2(3) = 2$

• Use induction to prove that the sequence $G_n(i)$, $i=0,\ldots,2^n-1$ is a binary-reflected Gray code.
Gray Code & Tower of Hanoi

- Introduce coordinates \((d_0, \ldots, d_{n-1})\), where \(d_i \in \{0, 1\}\)
- Associate \(d_i\) with the \(i\)th disk
- Initialize to \((0, \ldots, 0)\) and flip the \(i\)th coordinate when the \(i\)-th disk is moved
- The sequence of coordinate vectors obtained from the Tower of Hanoi solution is a Gray code (why?)
Tower of Hanoi

(0,0,0)
Tower of Hanoi

(0,0,1)
Tower of Hanoi

(0,1,1)
Tower of Hanoi

$(0, 1, 0)$
Tower of Hanoi

(1,1,0)
Tower of Hanoi

(1,1,1)
Tower of Hanoi

$(1, 0, 1)$
Tower of Hanoi

(1,0,0)
Hypercube

Graph (recursively defined)
n-dimensional cube has $2^n$ nodes with each node connected to $n$ vertices

Binary labels of adjacent nodes differ in one bit
Hypercube, Gray Code and Tower of Hanoi

A Hamiltonian path is a sequence of edges that visit each node exactly once.

A Hamiltonian path on a hypercube provides a Gray code (why?)