Convergence of Newton's Method

Note Title 11/17/2005

$$f(x) = 0$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Suppose $f(x_*) = 0$

When does $x_n \longrightarrow x_*$?

1. Find recurrence for the error

$e_k = x_k - x_*$. Will show

$|e_{k+1}| \leq C|e_k| \quad C < 1$

if x_0 is "close" ~~to~~ to x_*

relies on $f'(x_*) \neq 0$.

2. Use recurrence to prove convergence.

$$
\begin{aligned}
|e_{k+1}| &\le C\,|e_k| \\
&\le C^2\,|e_{k-1}| \\
&\;\;\vdots \\
&\le C^k\,|e_0| \longrightarrow 0
\end{aligned}
$$

Since $C < 1$, $C^k \to 0$.

1. Derive recurrence

$$\varphi(x) = x - \frac{f(x)}{f'(x)}$$

$$\varphi(x_n) = x_{n+1}$$

$$\varphi(x_*) = x_*$$

$\xi_k \in [x_k, x_*]$

$$e_{k+1} = x_{k+1} - x_*$$

$$= \varphi(x_k) - \varphi(x_*)$$

$$= \varphi'(\xi_k)\,(\underline{\underline{x_k - x_*}})$$

by Mean Value Theorem

Can choose x_0 sufficiently close to x_* such that

$$|\varphi'(\xi_k)| < 1$$

Since

$$\varphi'(x_*) = 0$$

$$\varphi(x) = x - \frac{f(x)}{f'(x)}$$

$$\varphi'(x) = 1 - \frac{f'(x)\,f'(x) - f(x)\,f''(x)}{f'(x)^2}$$

$$\Rightarrow \varphi'(x_*) = 0.$$

3. Rate of Convergence

Since $\varphi'(x_*) = 0$

$$\varphi(x_k) - \varphi(x_*) = \frac{\varphi''(\eta_k)}{2}(x_k - x_*)^2$$

$\eta_k \in [x_*, x_k]$ by Taylor Thm.

$$e_{k+1} = \frac{1}{2}\varphi''(\eta_k)\, e_k^2$$

Since $z_k \longrightarrow x_*$

(Since Newton converges $x_k \to x_*$

$\Rightarrow \quad z_k \to x_*$)

$$\lim_{k \to \infty} \frac{e_{k+1}}{e_k^2} = \frac{1}{2} \varphi''(x_*)$$

In other words

$$e_{k+1} \approx \frac{1}{2} \varphi''(x_*) e_k^2$$

i.e. quadratic convergence.

Exc. Show that

$$\frac{1}{2}\varphi''(x_*) = \frac{f''(x_*)}{2f'(x_*)}$$