**Symbolic Computation of Laplace-Dirichlet Eigenvalues**

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**Motivation**

In 2004, Grinfeld and Strang [4] posed the following boundary problem. What is the series in $i\Omega$ for the simple Laplace eigenvalues $\lambda_N$ on a regular polygon with $N$ sides under the Dirichlet boundary condition? The first four terms in the series were computed by hand in [5].

The function $w(r, \theta)$ is defined using Bessel Functions $J_i$ and the $i$th root of the zeroth Bessel Function $\rho$

$$w(r, \theta) = \frac{J_0(\rho r)}{\sqrt{\rho}} J_i(\rho)$$

(1)

The Laplacian eigenvalue is

$$\nabla^2 w = \lambda w \quad \text{(2)}$$

On the unit circle, under the Dirichlet boundary condition $w|_{\delta S} = 0,$ $\lambda_{\delta S} = \rho^2.$ To find $\lambda_N,$ we deform the circle into a polygon with $N$ sides and generate a Taylor Series. Solving for the fourth term required significant simplifications and the fifth term was intractable. This motivated the desire for a symbolic computation system that could handle this and similar problems.

**Model Problems**

To build a symbolic system, we selected model problems. A boundary variation of Poisson’s equation was used to justify automation was possible.[1] We then examined a variation on the polygon problem, deforming the circle into an ellipse with semi-axes $A = 1$ and $B = 1 + t$?

**Variations**

Each term in $\lambda_{\delta p e l l e}$ has the format $\int L_i dS$. The first variation $\lambda_1$ is defined by Hadamard’s formula.

$$\lambda_1 = - \int \nabla_i u \nabla^i w dS.$$  \quad \text{(4)}

The CMS can be used to find the next term in the series.

$$\lambda_2 = \frac{\delta}{\delta t} \left( - \int \nabla_i u \nabla^i w dS \right)$$

(5)

The CMS provides a recursive formula for these expressions.

$$\lambda_n = \int L_u dS, \lambda_n = \frac{\delta \lambda_{n-1}}{\delta t} - C B_n^m \lambda_{n-1}$$

(6)

Using this formula, we derive $L_2$

$$L_2 = - \frac{\delta \nabla_i u \nabla^i w}{\delta t} + C B_n^m \nabla_i u \nabla^i w$$

(7)

$L_2$ is reduced to its normal form by our TRS. Rules are applied to ensure an expression where evaluation is possible. Some key rules are shown.

Applied Product Rule:

$$\frac{\partial^n \nabla^n}{\partial t} \rightarrow - \frac{\partial^n}{\partial t} \nabla^n$$

(8)

Applied Chain Rule:

$$\frac{\partial^n \nabla^n}{\partial t} \rightarrow \nabla^n \frac{\partial^n}{\partial t} + C N^m \nabla^n \frac{\partial^n}{\partial t}$$

(9)

The final normal form for $L_2$ is

$L_2 = C^2 B_0 \nabla^i u \nabla_i C + C^2 B_1 \nabla^i u \nabla_i C - 2 C \nabla^i \frac{\partial^n}{\partial t} \nabla^n \frac{\partial^n}{\partial t}$

(10)

Using the recursive nature of the expression, further values can be calculated by the TRS.

**Calculus of Moving Surfaces**

The Calculus of Moving Surfaces (CMS) is an extension of Tensor Calculus to deforming manifolds. The CMS provides a mechanism for solving our problem by modeling the deformation of a circle into an ellipse. We can apply $\frac{\partial}{\partial t},$ the central operator of the CMS, to calculate each term in the series.

We automate the CMS using a Term Rewrite System (TRS) implementing equational properties as directional rules. The implementation of our TRS is described in [5]. After reduction by the TRS, expressions are evaluated by generating Maple code.

Some expressions are challenging to evaluate directly. The $\delta u/\delta t$-derivative of mean curvature $B_0^m$ can be replaced with an expression that is easier to evaluate $\nabla^i \nabla_i C + C B_0^m B_0^m.$ The $\delta u/\delta t$-derivative of $u(r, \theta)$ cannot be evaluated directly, replacing it with partial derivatives that we can calculate is crucial.

$$\delta u/\delta t = \frac{\partial u}{\partial t} + C N \nabla^i u$$

(3)

**Solutions**

We present the first five variations. They are provided in terms of $\lambda$ on the circle. We also provide the numeric approximation for each variation. The solutions have been confirmed to match numerical estimates.

$$\lambda = \lambda \quad \text{with } \lambda = 5.783$$

$$\lambda_1 = \frac{\lambda - \lambda^2}{2} \quad \text{with } \lambda = -5.783$$

$$\lambda_2 = \frac{\lambda - \lambda^2}{2} + \frac{\lambda^2}{3} \quad \text{with } \lambda = 17.036$$

$$\lambda_3 = \frac{\lambda - \lambda^2}{2} + \frac{\lambda^2}{3} \quad \text{with } \lambda = -67.517$$

$$\lambda_4 = \frac{\lambda - \lambda^2}{2} + \frac{\lambda^2}{3} \quad \text{with } \lambda = 333.919$$

$$\lambda_5 = \frac{\lambda - \lambda^2}{2} + \frac{\lambda^2}{3} \quad \text{with } \lambda = -1979.913$$

We give the first five terms in the series for the ellipse, in eccentricity $\epsilon$

$$\lambda(\epsilon) = \lambda - \frac{\lambda \epsilon^2}{2} + \left( \frac{\lambda}{3} - \frac{\lambda \epsilon^2}{2} \right) \epsilon^2 + \left( \frac{\lambda}{4} + \frac{\lambda \epsilon^2}{2} \right) \epsilon^3 + \left( \frac{\lambda}{5} - \frac{\lambda \epsilon^2}{2} \right) \epsilon^4$$

(11)

The first seven terms and details of calculation will be presented in [2]. In the CMS, an expression is true for all coordinate systems. This means that our expressions are also valid for the $N$-sided polygon problem. The values of $C$ and $\frac{\partial}{\partial t}$ are infinite Fourier series in the $N$-sided polygon problem.

**Conclusions**

We have automated the computation of CMS problems that has previously been solved by hand. We have extended the results on these problems past the point at which they had previously become intractable. We have shown that the CMS has potential for automation using a TRS. We have proven that our system produces accurate results by using it on problems that can also be solved by numerical approximation.

We will now apply our system to the $N$-sided polygon problem, which motivated its development. Due to the nature of the CMS, our derived expressions are true for any coordinate system. We must now solve the problem of evaluating these expressions on the $N$-sided polygon. The remaining hurdle of this problem is the complexity of evaluation on the $N$-sided polygon.

**References**


