Introduction to Combinatorial Game Theory

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Drexel MCS Society
April 10, 2008

A combinatorial game is a two-player game with the following properties:

- alternating play
- perfect information
- no element of chance
- guaranteed ending

A player left without a move loses the game.

Impartial games

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So. . . a game is uniquely determined by the positions to which it allows us to move.

The game of Nim

There are several heaps of sticks. A move consists of selecting a heap and removing one or more sticks from it. The winner is the player who takes the last stick.

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Note that each move affects exactly one heap.

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An impartial game G consists of a set of impartial games, called the options of G. A move in G consists of selecting one of its options.

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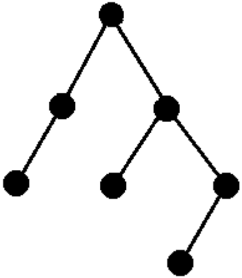
Consider. . . a tree is a set of trees.



We use a tree as an way of representing the games abstractly. We start at the root, and we play by moving from the root to one of its children.

Let's build some games

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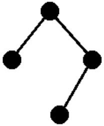
- B_2, the set containing these two: 

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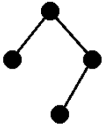
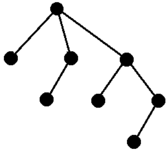
- B_2, the set containing these two: 
- B_3, the set containing these three: 

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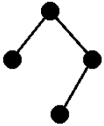

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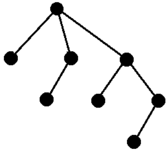
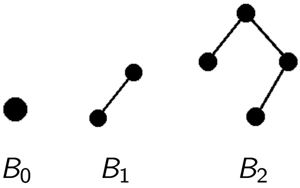


These are Nim-heaps: from B_k, one can move to $B_0, B_1, \ldots, B_{k-1}$.
These trees correspond to Nim-heaps of size 0, 1, 2, 3.

Who wins?



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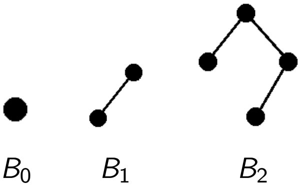

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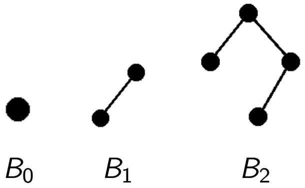

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You win by leaving your opponent a losing position.

Addition...

We model addition after Nim:

In $G \,\&\, H$, the player to move makes a move in either G or H, but not both.

Thus $G \,\&\, H = \{G \,\&\, x\}_{x \in H} \cup \{y \,\&\, H\}_{y \in G}$.

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B_0 is an identity element:

$B_0 \mathbin{\&} B_0 = B_0$,

and by induction:

$G \mathbin{\&} B_0 = \{x \mathbin{\&} B_0\}_{x \in G} = \{x\}_{x \in G} = G$ for any G.

We will use $\mathbf{0}$ to denote B_0.

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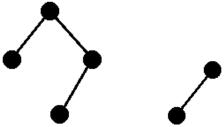

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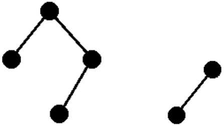

In the top sum, the first player loses; in the bottom sum, the first player wins!

Congruence

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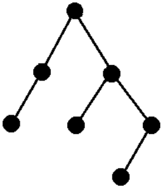
Two impartial games G and H are congruent iff for all games x, $G \,\&\, x$ and $H \,\&\, x$ have the same outcome.

Note that $G \cong G$ for all G.
But $G \cong H$ does not imply $G = H$. e.g. for all x,

 $\&\, x \quad \cong$  $\&\, x$.

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By induction, we assume that $G \,\&\, x = x \,\&\, G$ and $H \,\&\, y = y \,\&\, H$. Then the sets $G \,\&\, H$ and $H \,\&\, G$ are equal.

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By a similar argument, addition is also associative.

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PROOF. Given x, let $y = H \,\&\, x$. Then

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Thus $G_1 \,\&\, H \cong G_2 \,\&\, H$.

Corollary. If $G_1 \cong G_2$ and $H_1 \cong H_2$, then $G_1 \,\&\, H_1 \cong G_2 \,\&\, H_2$.

Identity

Adding a loss L to G does not change the outcome of G:

Every option of L is a win.
$G \,\&\, L = \{g \,\&\, L\}_{g \in G} \cup \{G \,\&\, \ell\}_{\ell \in L}$.

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Every loss is an identity element.
So all losses are congruent, and we obtain that

$$G \cong \mathbf{0} \text{ iff } G \text{ is a loss.}$$

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The negative of a number x is the y such that $x + y = 0$.
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PROOF. If G is a loss, it is apparent that $G \,\&\, G$ is a loss. Let us suppose then that G is a win.

Write $G = \{g_1, g_2, g_3, \ldots, g_n\}$. By induction, we assume that the theorem holds for $g_1, g_2, g_3, \ldots, g_n$.

$$
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Each game of the form $\{g_i \mathbin{\&} g_i, \ldots\}$ contains a loss, and thus is a win.

Consequently, each member of $G \mathbin{\&} G$ is a win, which makes $G \mathbin{\&} G$ a loss.

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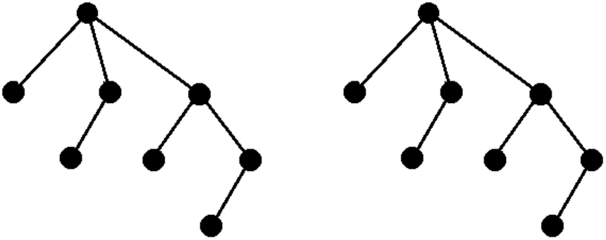
Therefore $G \mathbin{\&} G \cong \mathbf{0}$, q.e.d.

Negation, again

Consider playing the game $G \,\&\, G$. Suppose your opponent moves in the left subgame. What move should you play now?



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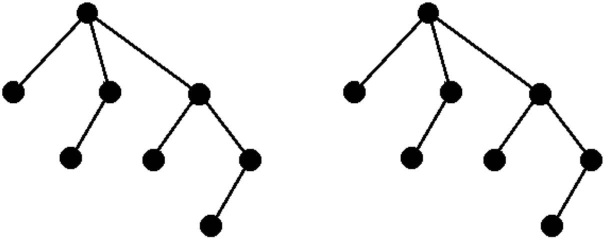

strategy-stealing

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THEOREM. If $G = \{g_1, g_2, \ldots, g_n\}$ and $H = \{h_1, h_2, \ldots, h_n\}$, with $g_1 \cong h_1$, $g_2 \cong h_2$, ..., $g_n \cong h_n$, then $G \cong H$. (Sets of congruent games are congruent.)

Proof.

$$
\begin{aligned}
G \mathbin{\&} H = \{ \quad & g_1 \mathbin{\&} H, g_2 \mathbin{\&} H, \ldots, g_n \mathbin{\&} H, \\
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Therefore $G \,\&\, H \cong \mathbf{0}$, and $G \cong H$, q.e.d.

Addition of Nim-heaps (Bouton, 1902)

THEOREM. When c is a power of 2 and $0 \leq d < c$, it holds that $B_c \,\&\, B_d \cong B_{c+d}$.

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Combined with the fact that $B_k \,\&\, B_k \cong \mathbf{0}$, we can use this theorem to add any two Nim-heaps:

e.g. $B_{22} \,\&\, B_5 \cong B_{16} \,\&\, B_4 \,\&\, B_2 \,\&\, B_4 \,\&\, B_1 \cong B_{16} \,\&\, B_2 \,\&\, B_1 \cong B_{19}$.

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Binary addition without carries: bitwise exclusive-OR

$$B_c \,\&\, B_d = B_{(c \texttt{ xor } d)}.$$

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&\quad \cup \{B_8 \,\&\, B_0, B_8 \,\&\, B_1, B_8 \,\&\, B_2\} \\
&\cong \{B_0, B_1, \ldots, B_7\} \cup \{B_8, B_9, B_{10}\} \\
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We can replace B_8 with B_c for any c that is a power of 2, and replace B_3 with B_d for $d < c$.

The Sprague-Grundy theorem (1930s)

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PROOF. By induction, we assume that the theorem holds true for $g_1, g_2, \ldots, g_n$.

Let k be the smallest integer ≥ 0 for which no element of G is congruent to B_k. We know that G contains elements congruent to each of $B_0, B_1, B_2, \ldots, B_{k-1}$.

$$
\begin{aligned}
G \mathbin{\&} B_k = \quad \{ \quad & g_1 \mathbin{\&} B_k, g_2 \mathbin{\&} B_k, g_3 \mathbin{\&} B_k, \ldots, g_n \mathbin{\&} B_k; \\
& G \mathbin{\&} B_0, G \mathbin{\&} B_1, G \mathbin{\&} B_2, \ldots, G \mathbin{\&} B_{k-1}\}.
\end{aligned}
$$

$G \mathbin{\&} B_0$ is a win since one of its options is $\cong B_0 \mathbin{\&} B_0$.

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$G \,\&\, B_0$ is a win since one of its options is $\cong B_0 \,\&\, B_0$.

$G \,\&\, B_1$ is a win since one of its options is $\cong B_1 \,\&\, B_1$, etc.

All the $g_i \,\&\, B_k$ are wins, because none of the g_i is congruent to B_k. ($B_i \,\&\, B_j \ncong \mathbf{0}$ for $i \neq j$)

$$G \mathbin{\&} B_k = \quad \{ \quad g_1 \mathbin{\&} B_k, g_2 \mathbin{\&} B_k, g_3 \mathbin{\&} B_k, \ldots, g_n \mathbin{\&} B_k; \\ G \mathbin{\&} B_0, G \mathbin{\&} B_1, G \mathbin{\&} B_2, \ldots, G \mathbin{\&} B_{k-1}\}.$$

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Every member of $G \mathbin{\&} B_k$ is a win; thus, $G \mathbin{\&} B_k$ is a loss and $\cong \mathbf{0}$, and we have $G \cong B_k$.

A game congruent to B_k is said to have a *Nim-value* of k and is denoted by $*k$.

Putting it all together

For impartial games, we have shown

- that congruent games behave the same;
- how to add Nim-heaps; and
- that every game is congruent to some Nim-heap.

We now know how to deal with any impartial game.

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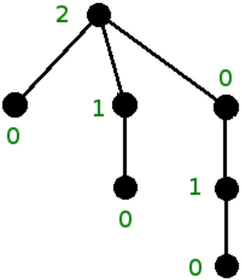
We now know how to deal with any impartial game.

Since congruent games are equivalent for our purposes, we will write $=$ in place of $\cong$ from now on.

Evaluation

The S-G theorem gives us a way to evaluate an arbitrary impartial game: just start at the lower levels of the tree and label each node with its Nim-value, deriving the Nim-values of branch nodes from their children.



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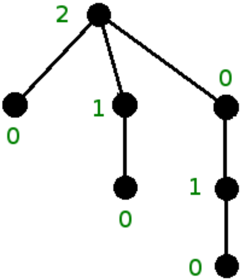

We can prune any tree into a binomial tree with the same Nim-value.

How to win at Nim

Losing positions have a Nim-value of 0; winning positions have a Nim-value > 0.
Remember that to win the game, we want to leave the opponent with a losing position.

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Take 2 from the pile of 5. This move wins because
$3 \texttt{ xor } 9 \texttt{ xor } 10 = 0$.

Limited Nim

Play is the same as in Nim, except that we only allow the player to take 1, 2, or 3 sticks at a time.

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$L_0 = \{\} = *0$
$L_1 = \{L_0\} = \{*0\} = *1$
$L_2 = \{L_0, L_1\} = \{*0, *1\} = *2$
$L_3 = \{L_0, L_1, L_2\} = \{*0, *1, *2\} = *3$

$L_4 = \{L_1, L_2, L_3\} = \{*1, *2, *3\} = *0$
$L_5 = \{L_2, L_3, L_4\} = \{*2, *3, *0\} = *1$
$L_6 = \{L_3, L_4, L_5\} = \{*3, *0, *1\} = *2$
$L_7 = \{L_4, L_5, L_6\} = \{*0, *1, *2\} = *3$

etc.

$$L_k = *(k \bmod 4).$$

Kayles

Bowling pins are set up in a row, with some gaps between them:

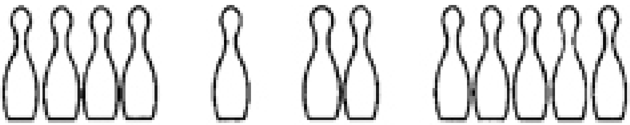

FIG. 26. The Kayles position $K_4 + K_1 + K_2 + K_5$.

Two bowlers compete to knock down the last pin. Each throws the ball perfectly, and can, at will, knock down a pin of his choosing, or knock down two adjacent pins.

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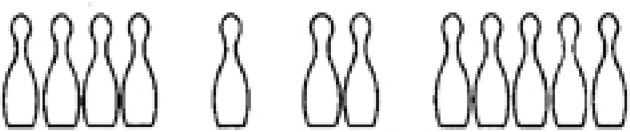

Fig. 26. The Kayles position $K_4 + K_1 + K_2 + K_5$.

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The sum of two Kayles games is another Kayles game, and each move affects only one of the clumps of pins. Therefore they behave as Nim-heaps:

Values of Kayles games

$$
\begin{aligned}
K_0 &= \{\} = *0 \\
K_1 &= \{K_0\} = \{*0\} = *1 \\
K_2 &= \{K_0, K_1\} = \{*0, *1\} = *2 \\
K_3 &= \{K_1, K_2, K_1 \,\&\, K_1\} = \{*1, *2, *0\} = *3 \\
K_4 &= \{K_2, K_3, K_1 \,\&\, K_2, K_1 \,\&\, K_1\} = \{*2, *3, *3, *0\} = *1 \\
K_5 &= \{K_3, K_4, K_2 \,\&\, K_2, K_1 \,\&\, K_1, K_1 \,\&\, K_3, K_2 \,\&\, K_2\} \\
&= \{*3, *1, *2, *0, *2, *0\} = *4
\end{aligned}
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etc.

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\end{aligned}
$$

etc.

Starting from K_{72}, the values settle into a repeating pattern with period 12:

$$*4, *1, *2, *8, *1, *4, *7, *2, *1, *8, *2, *7, \ldots$$

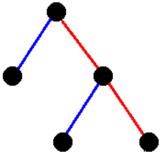
Extensions. . .

- misère play: the last player to move loses (e.g. misère Nim)
- partisan games — the two players have different options
- scoring — e.g. Go, Dots-and-Boxes

Partisan games (Berlekamp, Conway, and Guy, 1982)

What if we allow the two players to have different moves?

Let us call the two players Left and Right. A partisan game G consists of a pair of sets of games, the *left set* and the *right set*.



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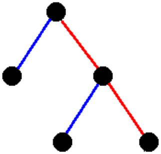

We can turn every impartial game into a partisan game.

Addition, negation

$x + y = \{x^L + y, x + y^L \mid x^R + y, x + y^R\}$.
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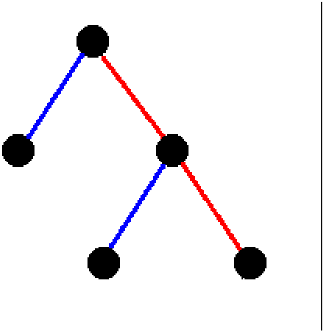

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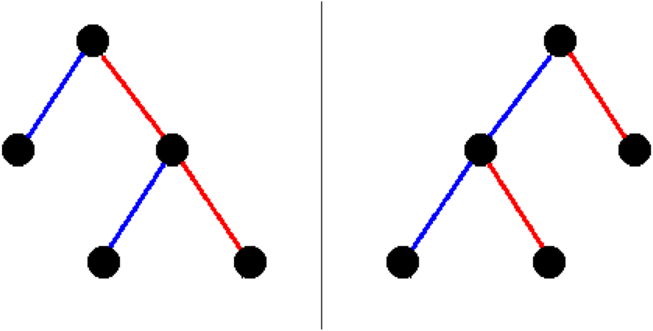

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To show $G = H$, we can show $G - H$ is a loss for the player to move.

To show $G > H$, we can show $G - H$ is a win for Left.

etc.

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To show $G > H$, we can show $G - H$ is a win for Left.

etc.

Formally,
$x \geq y$ iff no $x^R \leq y$ and $x \leq$ no y^L

Numbers (Conway, 1976)

The surreal numbers are those games

- formed from surreal numbers
- where no left option $\geq$ any right option.

For sets of numbers L and R, the number $\{L \mid R\}$ is the "simplest" number greater than L and less than R.

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For sets of numbers L and R, the number $\{L \mid R\}$ is the "simplest" number greater than L and less than R.

Neither player ever wants to move in a number unless it is the only move left.

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Let C_n be a left-leaning chain with n links. Then C_n behaves like the integer n. (Right-leaning is $-n$)

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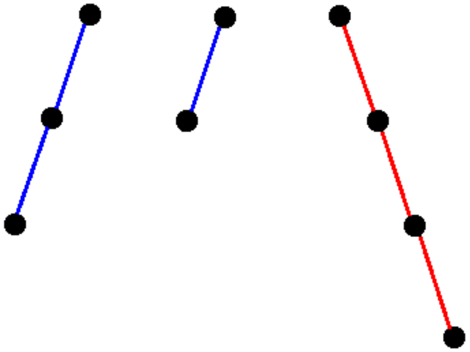

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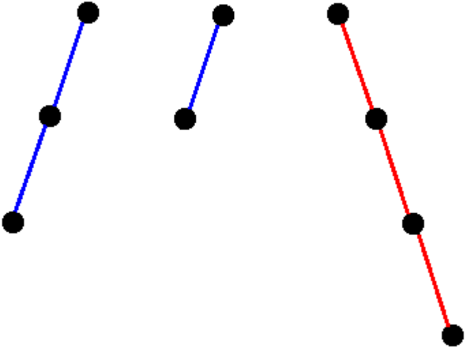
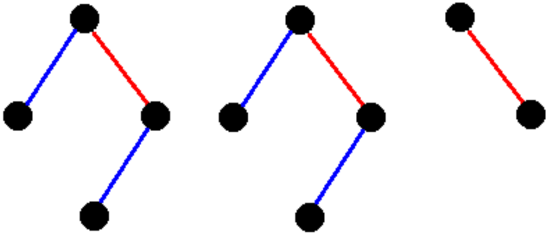


2 $+$ 1 - 3 $= 0$

Fractions

$$\{0 \mid 1\} = \frac{1}{2}:$$



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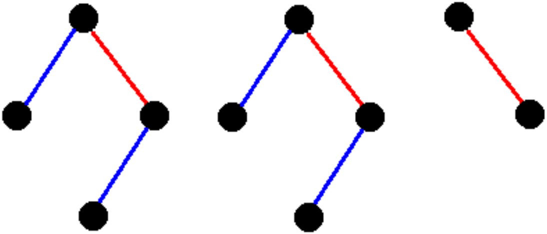

$x + x - 1 = 0$

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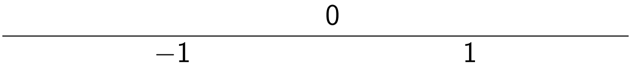

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$$\dots\dots\dots\dots\dots$$

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- Infinitesimals: $\frac{1}{\omega}, \ldots$

Infinity plus one!

$\omega = \{0, 1, 2, \dots \mid \}$.

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(1 is preferable to 0, 2 to 1, etc. And ω is preferable to any integer)

We have shown how games add. Games can also be split into sums. . .

(see e.g. Spight pg. 8; Berlekamp and Kim pg. 2, 3, 5)

Suggested reading

Claus Tøndering, *Surreal Numbers — An Introduction*:
http://www.tondering.dk/claus/surreal.html
Aaron Siegel, *Misère Games and Misère Quotients*:
http://arxiv.org/abs/math.CO/0612616

Games of No Chance, 1996:
http://www.msri.org/publications/books/Book29/contents.html

- Fraenkel, *Scenic Trails Ascending from Sea-Level Nim to Alpine Chess*
- West, *Championship-Level Play of Domineering*
- Elkies, *On Numbers and Endgames: Combinatorial Game Theory in Chess Endgames*
- Berlekamp and Kim, *Where Is the "Thousand-Dollar Ko"?*

More Games of No Chance, 2002:
http://www.msri.org/publications/books/Book42/contents.html

- Elkies, *Higher Nimbers in Pawn Endgames on Large Chessboards*
- Spight, *Go Thermography: The 4/21/98 Jiang–Rui Endgame*
- Berlekamp and Scott, *Forcing Your Opponent to Stay in Control of a Loony Dots-and-Boxes Endgame*
- Moore and Eppstein, *One-Dimensional Peg Solitaire, and Duotaire*

General references

- John H. Conway, *On Numbers and Games*. 1976, 2nd ed 2000.
- Berlekamp, Conway, and Guy, *Winning Ways for Your Mathematical Plays*. 1982, 2nd ed 2001-2004.
- Elwyn Berlekamp and David Wolfe, *Mathematical Go: Chilling Gets the Last Point*. 1994.
- Combinatorial Game Suite (software): `http://www.cgsuite.org/`

The material on impartial games comes from Chapters 6 and 16 of Conway; the material on partisan games comes mostly from Chapters 7 and 8 of the same.