Introduction to Counting

Definition: Pigeonhole Principle Given n objects with m *bins* to put these objects into, if $n > m$, then at least one of the bins must contain more than one of the given n objects.

Example A human head has between 0 and a 150,000. The city of London, England has several million citizens. Thus, at least two people living in london have the same exact number of hairs on their heads.

Definition: Permutation A permutation of a set $\{a_1, ...a_n\}$ is an ordering of the set written as $\langle a_{i_1}, ...a_{i_n}\rangle$. Here, the i_n subscripts are not necessarily in numerical order.

Example Given three people {Brandon, Maycock, Heather} the following is the set of all permutations of that set. We have the following permutations:

⟨ Brandon, Maycock, Heather ⟩
⟨ Brandon, Heather, Maycock ⟩
⟨ Maycock, Brandon, Heather ⟩
⟨ Maycock, Heather, Brandon ⟩
⟨ Heather, Brandon, Maycock ⟩
⟨ Heather, Maycock, Brandon ⟩

There are no other permutations of this set besides those listed.

Definiton: Factorial The factorial of a number $n > 0$ is n times the factorial of $n - 1$. The factorial of 0 is 1. The factorial function is notated with an exclaimation written after the given number. Thus, three factorial is written as $3!$, which is equivalent to $3*(3-1)! = 3*2! = 3*2*(2-1)! = 3*2*1! = 3*2*1*(1-1)! = 3*2*1*0! = 3*2*1*0 = 6$.

Theorem The number of permutations of a non-empty set with n elements is $n!$.

Proof Each ordering of the set represents an order in which we choose elements of the set, and each element is only chosen once (and every element is chosen). This is like having all the elements in a bag, and taking them out one after another until the bag is empty. When we start, there are n items in the bag, so there are n possibilities for the first element. After taking out the first item, there are $n - 1$ items in the bag, so there are $n - 1$ items that have the chance to be the second item. Continuing down, the last item chosen is the only option to choose, and then we are done. So we made n, $n - 1$, $n - 2$, ... 1 choices. Multiplying these all together, we get $n!$.

To see why we multiply the number of choices, consider this: If there are a choices for the first item, and b total permutations of the set without the first item, then there are $b + b + b + b$ (added a times) permutations to generate, depending on which of the a choices was taken. This amounts to $a * b$. Given the previous paragraph's argument, we conclude that the factorial function yields the proper number of permutations.

Problem: Cyclic Permutations If we consider two permutations to be the same if one can be gotten just by shifting elements either *left* or *right*, then how many permutations that would then be considered different are there? (Example: ⟨ Maycock, Brandon, Heather ⟩ and ⟨ Heather, Maycock, Brandon ⟩ are the same as all we did was shift everything to the *right* and bring the falling Heather back to the front).

Defintion: Variations The variations of size k on a set of size n are the permutations of subsets of k.

Example If we have a set $\{1, 2, 3, 4\}$, then the variations of size 2 are all the permutations of subsets of size 2. The subsets are the following:

$\{1, 2\}$
$\{1, 3\}$
$\{1, 4\}$
$\{2, 3\}$
$\{2, 4\}$
$\{3, 4\}$

So six total subsets all together. For each two element set, there are two permutations (either the bigger number is first or second). Thus, there are 12 total variations on the set.

Theorem For variations of size k on a set of size n, the total number of variations is $\frac{n!}{(n-k)!}$.

Proof For a given variation, we are choosing k elements in order out of n elements. This amounts to n choices for the first element, $n-1$ for the second, and all the way down to $n-k+1$ choices for the last element. Thus, we are choosing making n, $n-1$ ... $n-k+1$ choices, which is the product of the last k numbers less than or equal to n. Multiplying all the numbers from 1 to n together ($n!$), and dividing away the numbers we don't want yields the formula (taking away the first $n-k$ numbers leaves the last k numbers in the product).

Definition: Combinations The combinations of size k on a set of size n are the subsets of size k from the given set.

Note The combinations of a set are related to the variations in that we are taking the variations and disregarding order. So, two variations $\langle$ 1, 2 $\rangle$ and $\langle$ 2, 1 $\rangle$ are different (irrespective of the parent set), they are the same combination. Combinations are useful when we don't care about order (such as in poker. It doesn't matter what order you got the cards for making a royal flush are much less that you actually got it).

Theorem The number of combinations of a set is $\frac{n!}{k!*(n-k)!}$.

Proof Since there are $\frac{n!}{(n-k)!}$ variations, and noting that for each combination of size k there are $k!$ variations (as the variations that have the same elements amount to the permutations of the combination), dividing by $k!$ yields the proper formula.

Problem We're throwing a picnic, and picking up a total of 50 fruits and vegetables. There are 9 types that we can get – Green Peppers, Apples, Oranges, Bananas, Eggplants, Carrots, Corn, Tomatoes, Potatoes – and we have to get at least one of each. After taking one of each, we can get any number of the rest (including just all Peppers, splitting it as evenly as possible, etc. How many different arrangements of produce can we show up to the picnic with?