Introduction to Set Theory

Defintions

Definition: Set [Wiki] [Mathworld] A set is any collection of 'things.' Your immediate family is a set. A shopping list is a set of items that you wish to buy when you go to the store. The cars in the dealership parking lot is a set.

The only thing that matters to a set is what is in it. There is no notion of order or 'how many' of a particular item. A thing that is in a set is called an element or member of the set.

Definition: Empty Set [Wiki] [Mathworld] There is one particularly special set, the empty set, that contains no elements (it's empty).

Notation Membership in a set is usually denoted with a little backwards e. If S is a set, and x is some 'thing,' then we say x is an element of S by writing $x \in S$. When x is not in S, we write $x \notin S$.

Notation Sets are usually written out between curly braces. This notation can either explicitly list all the members of the set (i.e. $\{1, 2, 3\}$ being the set containing the first three positive integers), or can give a rule (i.e. $\{x|x \in \mathbb{N}, 2 \text{ divides } x\}$, where $\mathbb{N}$ are the natural numbers (0, 1, 2, ...), defines the even numbers. The bar means 'such that' so that the notation reads "The set of all x such that x is a natural number that is divisible by two").

The empty set is written as $\emptyset$, and with curly brace notation, is $\{\}$.

Example The following are all the same sets, due to the fact that there is no concept of multiplicity or order among sets.

{2, 3, 1}	{1, 1, 1, 2, 1, 3}
{1, 3, 2}	{2, 2, 1, 1, 3, 3}
{1, 2, 3}	{3, 2, 1}

All the above sets contain just the elements 1, 2, and 3, and that is all that matters.

Note As far as we're concerned, anything can be a member of a set. Another set can be a member of a set. It's generally frowned upon for a set to be a member of itself (this can lead to paradoxes, etc). So, we can have the set of types of numbers people learn about in school: $\{\mathbb{N}$ (the naturals), $\mathbb{Q}$ (the rationals), $\mathbb{R}$ (the reals), $\mathbb{C}$ (the complex numbers)$\}$. This is a set with four elements, each element being a set with an infinite number of members.

Definition: Subsets [Wiki] [Mathworld] For any given sets $\mathbb{A}$ and $\mathbb{B}$, $\mathbb{A}$ is a subset of $\mathbb{B}$ if every member of $\mathbb{A}$ is also a member of $\mathbb{B}$. An example are the naturals, $\mathbb{N}$, are a subset of the real numbers $\mathbb{R}$. If $\mathbb{A}$ is a subset of $\mathbb{B}$, but $\mathbb{B}$ contains elements not in $\mathbb{A}$, then we say that $\mathbb{A}$ is a proper subset of $\mathbb{B}$. Note that a set is always a subset of itself, and $\emptyset$, the empty set, is a subset of any set.

Notation If a set $\mathbb{A}$ is a subset of $\mathbb{B}$, we write $\mathbb{A} \subseteq \mathbb{B}$. If a set $\mathbb{A}$ is not a subset of $\mathbb{B}$, then we write $\mathbb{A} \nsubseteq \mathbb{B}$.

For $\mathbb{A}$ a proper subset of $\mathbb{B}$, we write $\mathbb{A} \subset \mathbb{B}$, and $\mathbb{A} \not\subset \mathbb{B}$ when $\mathbb{A}$ is not a proper subset of $\mathbb{B}$.

Similarly, one could write $\mathbb{B} \supseteq \mathbb{A}$, $\mathbb{B} \nsupseteq \mathbb{A}$, $\mathbb{B} \supset \mathbb{A}$, and $\mathbb{B} \not\supset \mathbb{A}$.

Definition: Superset A set $\mathbb{A}$ is a superset of a set $\mathbb{B}$ if $\mathbb{A} \supseteq \mathbb{B}$. A set $\mathbb{A}$ is a strict superset of $\mathbb{B}$ if $\mathbb{A} \supset \mathbb{B}$.

Definition: Set Equality Two sets are equal if they are both subsets of one another. The standard notation for equality, $=$, is used.

Definition: Powerset [Wiki] [Mathworld] The powerset of a set $\mathbb{A}$ is the set of all subsets of $\mathbb{A}$. Written in set notation, this is $\{x|x \subset \mathbb{A}\}$.

Notation The powerset of a set $\mathbb{A}$ is denoted as $\wp(\mathbb{A})$.

Operations

Definition: Intersection [Wiki] [Mathworld] For any two sets $\mathbb{A}$ and $\mathbb{B}$, define the intersection to the set of all elements of $\mathbb{A}$ and $\mathbb{B}$ that are in both sets. The set notation way to write this is $\{x|x \in \mathbb{A}$ and $x \in \mathbb{B}\}$ or $\{x|x \in \mathbb{A} \wedge x \in \mathbb{B}\}$, where $\wedge$ is logical notation for *and*.

Definition: Union [Wiki] [Mathworld] For any two sets $\mathbb{A}$ and $\mathbb{B}$, the union of $\mathbb{A}$ and $\mathbb{B}$ is the set containing all the elements of $\mathbb{A}$ and $\mathbb{B}$. The set notation way to write this is $\{x|x \in \mathbb{A}$ or $x \in \mathbb{B}\}$ or $\{x|x \in \mathbb{A} \vee x \in \mathbb{B}\}$, where $\vee$ is the logical notation for *or*.

Definition: Product [Wiki] [Mathworld] The product of two sets $\mathbb{A}$ and $\mathbb{B}$ is the set of ordered pairs (a, b) where $a \in \mathbb{A}$ and $b \in \mathbb{B}$. Note that there is also the reverse product of (b, a), but these are different sets. Also note that *every* possible combination of a and b are in the product.

Notation For any two sets $\mathbb{A}$ and $\mathbb{B}$, the following notations can be made:

- $\mathbb{A} \cap \mathbb{B}$ for intersection of $\mathbb{A}$ and $\mathbb{B}$.
- $\mathbb{A} \cup \mathbb{B}$ for the union of $\mathbb{A}$ and $\mathbb{B}$.
- $\mathbb{A} \times \mathbb{B}$ for the product of $\mathbb{A}$ and $\mathbb{B}$.

Definition: Set Subtraction [Wiki] [Mathworld] The subtraction of one set $\mathbb{B}$ from another set $\mathbb{A}$ is defined to be the set of all elements of $\mathbb{A}$ that are not in $\mathbb{B}$. This is also called the complement of $\mathbb{B}$ in $\mathbb{A}$.

Notation The set subtraction of one set $\mathbb{B}$ from $\mathbb{A}$ is written as $\mathbb{A} \setminus \mathbb{B}$.

Relations

Definition: Relation [Wiki] [Mathworld] For any two sets $\mathbb{A}$ and $\mathbb{B}$, a relation on the sets $\mathbb{A}$ and $\mathbb{B}$ is a subset of $\mathbb{A} \times \mathbb{B}$. The relation may not contain any elements of the product of $\mathbb{A}$ and $\mathbb{B}$. Such a relation is called the empty relation.

Notation For a relation $\mathbb{R}$, $(a, b) \in \mathbb{R}$ can be written as $a\mathbb{R}b$.

Definition: Relation on a Set A relation $\mathbb{R}$ on a set $\mathbb{A}$ is a relation on the product $\mathbb{A} \times \mathbb{A}$.

Definition: Reflexive [Wiki] [Mathworld] A relation $\mathbb{R}$ on a set $\mathbb{A}$ is reflexive if for all $a \in \mathbb{A}$, $a\mathbb{R}a$ (i.e. $(a, a) \in \mathbb{R}$).

Definition: Symmetric [Wiki] [Mathworld] A relation $\mathbb{R}$ on set is symmetric if whenever $(a, b) \in \mathbb{R}$ is true, $(b, a) \in \mathbb{R}$ is also true.

Definition: Transitive [Wiki] [Mathworld] A relation $\mathbb{R}$ on a set is transitive if whenever $(a, b) \in \mathbb{R}$ and $(b, c) \in \mathbb{R}$, then $(a, c) \in \mathbb{R}$.

Definition: Equivalence Relation [Wiki] [Mathworld] A relation on a set is an equivalence relation if it is reflexive, symmetric, and transitive.

Definition: Equivalence Class [Wiki] [Mathworld] Given a set $\mathbb{A}$ and an equivalence relation $\mathbb{E}$ on $\mathbb{A}$, the equivalence class of any member a of $\mathbb{A}$ is the set of all elements of $\mathbb{A}$ that are related to a by the equivalence relation. The set notation way to write this for one specific a is $\{x|x \in \mathbb{A}, a\mathbb{E}x\}$.

Notation The equivalence class of an element x of a set may be written as $[x]$. The equivalence relation and set involved should be understood given the context.

Theorem For any two equivalence classes $[x]$ and $[y]$ of an equivalence relation $\mathbb{E}$ on a set $\mathbb{A}$, either $[x] \cap [y] = \emptyset$ or $[x] = [y]$.

Proof Assume that there are two equivalence classes that do not follow the theorem – i.e. there are an $[x]$ and $[y]$ such that $[x] \neq [y]$ and $[x] \cap [y] \neq \emptyset$. Then, since $[x]$ and $[y]$ are not the same *and* there are elements of $[x]$ that are in $[y]$, we can let w be such an element. Then, for any $s \in [x]$, $s\mathbb{E}w$. Also, for any $t \in [y]$, $w\mathbb{E}t$. By transitivity, $s\mathbb{E}t$, and thus every element of $[x]$ is related to $[y]$ by $\mathbb{E}$. Thus, by the definition of equivalence class, $[x] = [y]$.

The above shows that an equivalence relation partitions a set into disjoint sets which we call equivalence classes. The converse is also true: any partition of a set forms an equivalence relation on that set.

Functions

Definition: Function [Wiki] [Mathworld] A function is a relation $\mathbb{F}$ drawn from $\mathbb{A} \times \mathbb{B}$ such that for every $a \in \mathbb{A}$, there is exactly one $b \in \mathbb{B}$ such that $(a, b) \in \mathbb{F}$. (i.e. every element of $\mathbb{A}$ is in the function, and every element of $\mathbb{A}$ only gets ‘assigned’ to one element of $\mathbb{B}$.) The set $\mathbb{A}$ is called the domain of $\mathbb{F}$, and the set $\mathbb{B}$ is called the codomain of $\mathbb{F}$.

For a given element x of the domain of a function, the corresponding element of the codomain is called the image of x.

Notation Given the sets $\mathbb{A}$ and $\mathbb{B}$ with the function $\mathbb{F}$ on $\mathbb{A} \times \mathbb{B}$, $\mathbb{F}$ may be described with $\mathbb{F} : \mathbb{A} \mapsto \mathbb{B}$. Note that the 'set font' is usually dropped for the function, which would now be written as $f : \mathbb{A} \mapsto \mathbb{B}$.

Sometimes, the familiar function notation is used (in fact, usually it is). This is $f(x) = y$ to state that y is the image of x under f.

Definition: Range [Wiki] [Mathworld] The range of a function f is the set of all elements of the codomain that belong to some pair existing in the function. (i.e. $b \in \mathbb{B}$ is in the range of $f : \mathbb{A} \mapsto \mathbb{B}$ if $(a, b) \in f$ for some $a \in \mathbb{A}$.

Note Not every element of the codomain will appear in the range. If we have a function $f : \mathbb{N} \mapsto \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, ...\}$ defined as $f(x) = 3$, then the range is $\{3\}$, while the codomain is the natural numbers.

Definition: One-to-One $(1 - 1)$ [Wiki] [Mathworld] A function is one-to-one if for any element of the range, there is exactly one element of the domain that is mapped to it – i.e. for a function $f : \mathbb{A} \mapsto \mathbb{B}$, $x \in \mathbb{A}$, $y \in \mathbb{B}$, $x \neq y$, then $f(x) \neq f(y)$ if f is one-to-one. A one-to-one function is also called an injection.

Definition: Onto [Wiki] [Mathworld] A function is onto if the range of the function is equal to the codomain of the function – i.e. for a function $f : \mathbb{A} \mapsto \mathbb{B}$ every $b \in \mathbb{B}$ is the result of $f(a)$ for some $a \in \mathbb{A}$. An onto function is also called a surjection.

Definition: Bijection [Wiki] [Mathworld] A function is a bijection if it is both onto and one-to-one.

Definition: Inverse Function [Wiki] [Mathworld] For a given function $f : \mathbb{A} \mapsto \mathbb{B}$, the inverse function f^{-1} is defined as $\{(b, a) | (a, b) \in f\}$. The inverse is only defined for bijections (all other functions inverses would violate one of the two rules defining functions). The inverse of a bijection is again a bijection.

Definition: Composition [Wiki] [Mathworld] For two functions $f : \mathbb{A} \mapsto \mathbb{B}$ and $g : \mathbb{B} \mapsto \mathbb{C}$, the composition of f and g is the function $h : \mathbb{A} \mapsto \mathbb{C}$ such that $h(x) = g(f(x))$.

Notation The composition of two functions f and g can be written as $g \circ f$, read as 'g of f.' and is eqivalent to the function $h(x) = g(f(x))$ for the appropriate domains and codomains.

Theorem The composition of two one-to-one functions is again one-to-one.

Proof Let $f : \mathbb{A} \mapsto \mathbb{B}$ and $g : \mathbb{B} \mapsto$ be two such functions. Showing that $a \neq b$ implies that $(g \circ f)(a) \neq (g \circ f)(b)$ will imply that $(g \circ f)$ is one-to-one.

$(g \circ f)(a)$ and $(g \circ f)(b)$ are $g(f(a))$ and $g(f(b))$, respectively. Since f is one-to-one, we know that $f(a) \neq f(b)$ since $a \neq b$. Since g is one-to-one, and $f(a) \neq f(b)$, we now know that $g(f(a)) \neq g(f(b))$ is true. This is exactly what we wanted to prove.

Theorem For any two onto functions, their composition is onto.

Proof Let $f : \mathbb{A} \mapsto \mathbb{B}$ and $g : \mathbb{B} \mapsto \mathbb{C}$ be two such functions. Their composition, $g \circ f : \mathbb{A} \mapsto \mathbb{C}$, has a codomain $\mathbb{C}$. Take any $c \in \mathbb{C}$. Then, $g(b) = c$ for some $b \in \mathbb{B}$, because g is onto. Then, $f(a) = b$ for some $a \in \mathbb{A}$, because f is onto. Then, $(g \circ f)(a) = c$, and our arbitrary c is the image of a under the function $g \circ f$. Thus, every element of the codomain is mapped to by the domain, and is thus onto.

Theorem The composition of two bijections is another bijection.

Proof This is the combination of the previous two theorems.

Cardinality

Definition: Cardinality [Wiki] [Wiki] [Mathworld] Two sets are said to have the same cardinality if there exists a bijection between them. If, for two given sets $\mathbb{A}$ and $\mathbb{B}$, there exists no bijection between them, but there exists a bijection between $\mathbb{A}$ and a subset of $\mathbb{B}$, then we say that $\mathbb{A}$ is smaller than $\mathbb{B}$.

A cardinal number is either an a finite number describing the size of a set (all sets of the same size have the same cardinality), or an infinite cardinal (described more below). Two sets that have the same cardinality have a bijection between them. The reason this is used as the definition of "size" in set theory is that it's a very trivial way of 'counting' sizes without actually counting. We take two sets, and if for every element of one set we have exactly one element of the other set, and vice versa, then we have the same number of elements in both sets (otherwise, some would be left in one of the sets after we're done pairing).

Notation For a given set $\mathbb{A}$, the cardinality is written as $|\mathbb{A}|$.

Definition: Infinite Sets (Transfinite Set) [Wiki] [Mathworld] A set is said to be infinite if it has the same cardinality as a proper subset of itself. Otherwise, it is said to be finite and its cardinality can be refered to by the natural number corresponding to the number of elements it possesses.

Definition: Aleph Null [Wiki] [Mathworld] The cardinality of the $\mathbb{N}$, the set of natural numbers, is called Aleph Null, and is written as $\aleph_0$.

Definition: Countable and Uncountable Sets [Wiki] [Wiki] [Mathworld] [Mathworld] [Mathworld] A set is countable if it has cardinality $\aleph_0$ or is finite. A set that is not countable is said to be uncountable.

Note If two sets have the same cardinality - i.e. there exists a bijection between them - then those two sets have the same number of elements. Cardinality is another word for 'size' that allows for comparing infinitely large sets.

Notation We use the following to denote various sets:

- $\mathbb{N}$ is the set of natural numbers $\{0, 1, 2, ...\}$.
- $2\mathbb{N}$ is the set of even natural numbers $\{0, 2, 4, ...\}$.
- $2\mathbb{N} + 1$ is the set of odd natural numbers $\{1, 3, 5, ...\}$.
- $-\mathbb{N}$ is the set of negative counting numbers from -1 down $\{-1, -2, -3, ...\}$.
- $\mathbb{Z}$ is the set of integers $\{..., -2, -1, 0, 1, 2, ...\}$.
- $\mathbb{Q}$ is the set of rational numbers $\{\frac{-1}{4}, 0, 4, \frac{8}{9}, ...\}$.
- $\mathbb{R}$ is the set of real numbers $\{\pi, 0, \sqrt{2}, ...\}$.

Theorem There are just as many even natural numbers as there are natural numbers.

Proof We define the function $f : \mathbb{N} \mapsto 2\mathbb{N}$ as follows: $f(x) = 2x$. The fact that every even natural gets mapped is clear. Such a number, being even, would be of the form $2x$, and thus would get mapped to by x. The fact that it is one-to-one follows from the fact that if $f(x) = f(y)$, then we'd have $2x = 2y$, and dividing by 2 shows that $x = y$, so only one natural maps to any given even natural.

Theorem There are just as many integers (the set containing zero, the positive counting numbers, and the negative counting numbers) as there are naturals.

Proof We define two functions: $p : 2\mathbb{N} \mapsto \mathbb{N}$ and $n : 2\mathbb{N} + 1 \mapsto -\mathbb{N}$ that are both bijections.

p is given by the rule $p(x) = x/2$. This is the inverse of the bijection used to map the naturals, $\mathbb{N}$, to the evens, $2\mathbb{N}$. Thus, p is a bijection.

n is given by the rule $n(x) = -(x+1)/2$. This function maps every odd number to a negative number. To see that it is one to one is a matter of simple arithmetic (assuming different x values and then cancelling, as before). To show that it is onto, take any negative number y and solve for x with $y = -(x + 1)/2$ and we see that $-(2y + 1) = x$, which is an odd natural number.

Now, take the two functions p and n and make the function $z : \mathbb{N} \mapsto \mathbb{Z}$ as follows: $z(x) = p(x)$ if x is even, and $z(x) = n(x)$ if x is odd. There are three things that we have to show with z:

- That it is in fact a function
- That it is onto
- That it is one-to-one

To show that it is a function, we need to show that every element of the domain, $\mathbb{N}$ gets mapped to exactly one element of the codomain. This is obvious as whether a given member of $\mathbb{N}$ is even or odd, we assign its image to exactly one positive or negative a member of $\mathbb{Z}$, respectively.

To show that z is onto, note that for any negative member of $\mathbb{Z}$, an odd member of $\mathbb{N}$ will be mapped to it. For any *other* member of $\mathbb{Z}$, an even member of $\mathbb{N}$ will be mapped to it.

To see that it is one-to-one, note that for any distinct $a, b \in \mathbb{N}$, either both are even, both are odd, or there is one of each. If they are neither both even or odd, then one will be mapped to a negative integer, and the other to a nonnegative integer. If both are even or both are odd, then their images will be different because the functions p and n are one-to-one.

Thus, $\mathbb{Z}$, the set of integers, has the same cardinality as $\mathbb{N}$, the set of natural numbers.

Theorem Show that for a countable set $\mathbb{A}$, $\mathbb{A} \times \mathbb{A}$ is countable.

Proof Let $g : \mathbb{N} \mapsto \mathbb{A}$ be a bijection between $\mathbb{N}$ and $\mathbb{N}$. From now on, we shall use $\mathbb{N}$ instead of the set $\mathbb{N}$, as we can use g and g^{-1} to relabel them back and forth.

If we arrange our elements in a table where both axes are indexed by $\mathbb{N}$,start at 0 and are increasing, we can let the elements of the table correspond to the horizontal and vertical index at which it sits. Note that the first diagonal would contain the element $(0,0)$ and only $(0,0)$. The second diagonal would contain only $(0,1)$ and $(1,0)$. And so on, and so on. Each diagonal contains one more element than the last, and each table element is on a diagonal. Enumerating through the diagonal elements from smallest to largest in the first coordinate yields a method of each element of the table once and only once. This forms a function from the set $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$. Relabelling the table elements to $\mathbb{A}$ completes the proof.

Theorem Any subset of a countable set is countable.

Proof Let $\mathbb{A}$ a countable set and $\mathbb{B} \subset \mathbb{A}$ any subset of $\mathbb{A}$. If $\mathbb{B}$ is finite, then it is countable by definition. If it is infinite, then let $g : \mathbb{N} \mapsto \mathbb{A}$ be a bijection between the naturals and $\mathbb{A}$, the superset of $\mathbb{B}$. Define a function $f : \mathbb{N} \mapsto \mathbb{B}$ such that the i^{th} element of $\mathbb{N}$ is mapped to the i^{th} smallest element of $\mathbb{B}$, ordering the elements of $\mathbb{B}$ as the naturals under the function g. Every element of $\mathbb{B}$ will be mapped, and never by more than one natural number.

Theorem The set $\mathbb{Q}$ of rational numbers is countable.

Proof Since each $r \in \mathbb{Q}$ can be written as a pair of integers (such as $\frac{2}{4}$ as $(2,4)$). Thus, $\mathbb{Q}$ is a subset of $\mathbb{Z} \times \mathbb{Z}$, which is countable, and thus $\mathbb{Q}$ is countable.

Theorem [Wiki] [Mathworld] The set of all real numbers between zero and one, including zero and one, is uncountable.

Proof In this proof, we shall use the the set $1, 2, 3, ...$ as the naturals instead of $0, 1, 2, 3, ...$ The sets are obviously 'the same' with 0 relabelled 1, etc, etc – and this is just a trick to make notation easier.

Assume that $[0,1]$ (the closed interval between zero and one) is indeed countable. Then, there exists a bijection $f : \mathbb{N} \mapsto [0,1]$. Let $z = 0.d_1d_2d_3...$ be a real number where each d_i is the i^{th} digit after the decimal. Now, for each $i \in \mathbb{N}$, there is a real number $x \in [0,1]$ such that $x = f(i)$. If the i^{th} digit of x after the decimal point is a 3, then let the digit d_i in z be a 4, otherwise let it be a 3. The number z is in the set $[0,1]$ but cannot possibly be in the range of f, as it differs from every element in the range of f in at least one decimal place. Thus, no such f can exist. Thus, $[0,1]$ is an infinite set larger than $\mathbb{N}$, and is thus uncountable.

Theorem [Wiki] [Mathworld] There exists no bijection between a set and its powerset (i.e. there is no largest set).

Proof Assume that there does exist a set $\mathbb{A}$ such that $|\mathbb{A}| = |\wp(\mathbb{A})|$, with $f : \mathbb{A} \mapsto \wp(\mathbb{A})$ being a bijection between them. Then, any set whose members are drawn from $\mathbb{A}$ will be a subset of $\mathbb{A}$ and thus an element of $\wp(\mathbb{A})$. Let $\mathbb{T} = \{x | x \in \mathbb{A},\ x \notin f(x)\}$, the set of all elements of $\mathbb{A}$ that are not in the powerset element they get mapped to by f. $\mathbb{T}$ is a subset of $\mathbb{A}$, thus $\mathbb{T} \in \wp(\mathbb{A})$. Since f is a bijection, every element of $\wp(\mathbb{A})$ is in the range of f, and there must be some $y \in \mathbb{A}$ such that $\mathbb{T} = f(y)$. From this, a contradiction arises. Either $y \in \mathbb{T}$, in which case y cannot be in $\mathbb{T}$ due to the definition of $\mathbb{T}$, or y is not a member of $\mathbb{T}$, in which case it must be a member of $\mathbb{T}$, again due to $\mathbb{T}$'s definition. Thus, no such bijection f can exist.

Theorem For any finite set $\mathbb{A}$, $|\wp(\mathbb{A})| = 2^{|\mathbb{A}|}$.

Proof For every $x \in \mathbb{A}$, either $x \in \mathbb{B}$ or $x \notin \mathbb{B}$ for any $\mathbb{B} \subset \mathbb{A}$. Since there are two options for any such x and $\mathbb{B}$, there are $2 * 2 * 2 * 2...2$ (repeated $|\mathbb{A}|$ times) ways to make a subset of $\mathbb{A}$. Thus, there are $2^{|\mathbb{A}|}$.

Definition: Finite Addition If $\mathbb{A}$ and $\mathbb{B}$ are finite, then $|\mathbb{A}| + |\mathbb{B}|$ is the finite number equal to the sum of the cardinalities.

Definition: Transfinite Arithmetic If $\mathbb{A}$ or $\mathbb{B}$ are infinite, then $|\mathbb{A}| + |\mathbb{B}| = max(|\mathbb{A}|, |\mathbb{B}|)$.

Theorem The cardinality of the union of two finite sets $\mathbb{A}$ and $\mathbb{B}$ is given by the following formula: $|\mathbb{A} \cup \mathbb{B}| = |\mathbb{A}| + |\mathbb{B}| - |\mathbb{A} \cap \mathbb{B}|$.

Proof Counting each element of $\mathbb{A}$ and adding for the number of elements of $\mathbb{B}$ yeilds $|\mathbb{A}| + |\mathbb{B}|$. Each element of $\mathbb{A} \setminus \mathbb{B}$ is counted once, as is each element of $\mathbb{B} \setminus \mathbb{A}$. The only elements left after these two sets are the elements of $\mathbb{A} \cap \mathbb{B}$, but they would all be counted twice, once for being in $\mathbb{A}$ and once for being in $\mathbb{B}$. Thus, $|\mathbb{A} \cup \mathbb{B}| = |\mathbb{A}| + |\mathbb{B}| - |\mathbb{A} \cap \mathbb{B}|$.